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James Richards

~~James Richards~~

# ARITHMETIC,

OR,

## PART I. OF VOL. I.

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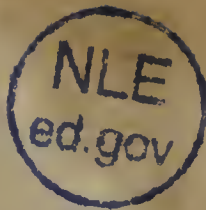
# COURSE

OF

# MATHEMATICS,

IN TWO VOLUMES:

COMPOSED, AND MORE ESPECIALLY DESIGNED,

*FOR THE USE OF THE GENTLEMEN CADETS*

*IN THE ROYAL MILITARY ACADEMY AT WOOLWICH.*

BY

*CHARLES HUTTON, LL.D., F.R.S.*

AND PROFESSOR OF MATHEMATICS IN THE SAID ACADEMY.

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THE THIRD EDITION,

*ENLARGED AND CORRECTED.*

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VOL. I.

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# PREFACE.

A SHORT and Eafy Courfe of the Mathematical Sciences has long been confidered as a defideratum for the ufe of Students in the different fchools of education : one that fhould hold a middle rank between the more voluminous and bulky collections of this kind, and the mere abftract and brief common-place forms, of principles and memorandums.

For long experience in all Seminaries of Learning, and particularly in the Royal Military Academy at Woolwich, has fhewn, that fuch a work was very much wanted, and would prove a great and general benefit ; as, for want of it, recourfe has always been obliged to be had to a number of other books, of different authors, felecting a part from one and a part from another, as feemed moft fuitable to the purpofe in hand, and rejecting the other parts : a practice which occafions much expence and trouble, in procuring and keeping fuch a number of odd volumes, of various modes of compofition and form ; befides wanting the benefit of uniformity and reference, which are found in a regular feries of compofition.

To remove thefe inconveniences, the Author of the prefent work has been induced, from time to time, to compofe various parts of this Courfe of Mathematics ; which the experience of many years ufe in the Academy has enabled him to adapt and improve to the moft ufeful form and quantity, for the benefit of inftruction. And, to render that benefit more eminent and lafting, the Mafter General of the Ordnance has been pleafed to give it its prefent form, by ordering it to be enlarged and printed.

As this work has been compofed exprefsly with the intention of adapting it to the purpofes of academical education, it is not defigned to hold out the expectation of new

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inventions or difcoveries : but rather to collect and arrange the moft ufeful principles in a convenient practical form, demonftrate them in a plain and concife way, and illuftrate them with fuitable examples : rejecting whatever feem to be matters of mere curiofity ; and retaining only fuch parts and branches, as have a direct tendency and application to fome ufeful purpofe in life, efpecially in the military profeffion, for which the gentlemen educated at this Academy are intended.

As a work of fuch a nature muft neceffarily confift of matters which have, in a manner, become common property, and in a great meafure are contained, in fome fhape or other, in moft books of this kind, it will not be imputed to the author, as a crime, that he has availed himfelf of the materials of fome of the beft books on thefe fciences, from whence he may have extracted, or which he may have imitated ; whether they be any of his own former publications, or thofe of other authors.

Neverthelefs it is expected that fomething new may be found in many parts of thefe volumes, as well in the matter, as in the arrangement and manner of demonftration, efpecially in the geometrical part of this work. And here the author hopes he will not be too feverely criticifed if, through a defire of rendering this branch more eafy and fimple, he has in fome inftances deviated a little from the tedious and rigid ftrictnefs of Euclid, particularly in the doctrine of ratios and proportion, which has always been fo greatly complained of, efpecially by young ftudents in thefe fciences.

*Royal Military Academy,*  
 *July* 30*th*, 1798. }

Befides recomputing the examples, and rendering them more correct in the numbers, this edition is much enlarged in feveral places, and particularly by extending the tables of fquares and cubes, fquare roots and cube roots, to upwards of 1000 numbers, which will be found of great ufe in many calculations.

3

*Oct.* 10, 1800.

# CONTENTS OF PART I.
## VOL. I.

*Division*

A COURSE

# A

# COURSE

## OF

# MATHEMATICS, &c.

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# GENERAL PRINCIPLES.

1. QUANTITY, or MAGNITUDE is any thing that will admit of increafe or decreafe; or that is capable of any fort of calculation or menfuration : fuch as, numbers, lines, fpace, time, motion, weight.

2. MATHEMATICS is the fcience which treats of all kinds of quantity whatever, that can be numbered or meafured.— That part which treats of numbering, is called *Arithmetic:* and that which concerns meafuring, or figured extenfion, is called *Geometry.*——Thefe two, which are converfant about multitude and magnitude, and are the foundation of all the other parts, are called *Pure* or *Abftract Mathematics ;* becaufe they inveftigate and demonftrate the properties of abftract numbers and magnitudes of all forts. And when thefe two parts are applied to particular or practical fubjects, they conftitute the branches or parts called *Mixed Mathematics.*—— Mathematics is alfo diftinguifhed into *Speculative* and *Practical ;* viz. *Speculative,* when it is concerned in difcovering properties and relations ; and *Practical,* when applied to practice and real ufe concerning phyfical objects.

3. In Mathematics are feveral general terms or principles; fuch as, Definitions, Axioms, Propofitions, Theorems, Problems, Lemmas, Corollaries, Scholiums, &c.

4. A *Definition* is the explication of any term or word in a fcience; fhewing the fenfe and meaning in which the term is employed.—Every Definition ought to be clear, and expreffed in words that are common and perfectly well underftood.

5. A *Propofition* is fomething propofed to be proved, or fomething required to be done; and is accordingly either a Theorem or a Problem.

6. A *Theorem* is a demonftrative propofition; in which fome property is afferted, and the truth of it required to be proved. Thus, when it is faid that, The fum of the three angles of any triangle, is equal to two right angles, this is a Theorem, the truth of which is demonftrated by Geometry. —A fet or collection of fuch Theorems conftitutes a *Theory*.

7. A *Problem* is a propofition or a queftion requiring fomething to be done; either to inveftigate fome truth or property, or to perform fome operation. As, to find out the quantity or fum of all the three angles of any triangle, or to draw one line perpendicular to another.——A *Limited Problem* is that which has but one anfwer or folution. *An Unlimited Problem* is that which has innumerable anfwers. And a *Determinate Problem* is that which has a certain number of anfwers.

8. *Solution* of a Problem, is the refolution or anfwer given to it. A *Numerical* or *Numeral Solution*, is the anfwer given in numbers. A *Geometrical Solution*, is the anfwer given by the principles of Geometry. And a *Mechanical Solution*, is one which is gained by trials.

9. A *Lemma* is a preparatory propofition, laid down in order to fhorten the demonftration of the main propofition which follows it.

10. A *Corollary*, or *Confectary*, is a confequence drawn immediately from fome propofition or other premifes.

11. A *Scholium*, is a remark or obfervation made on fome foregoing propofition or premifes.

12. An *Axiom*, or *Maxim*, is a felf-evident propofition; requiring no formal demonftration to prove the truth of it; but is received and affented to as foon as mentioned. Such as, The whole of any thing is greater than a part of it: or, The whole is equal to all its parts taken together: or, Two quantities that are each of them equal to a third quantity, are equal to each other.

13. A *Poftu-*

13. *A Poftulate*, or *Petition*, is fomething required to be done, which is fo eafy and evident that no perfon will hefitate to allow it.

14. *An Hypothefis* is a fuppofition affumed to be true, in order to argue from, or to found upon it the reafoning and demonftration of fome propofition.

15. *Demonftration*, is the collecting the feveral arguments and proofs, and laying them together in proper order, to fhew the truth of the propofition under confideration.

16. *A Direct*, *Pofitive*, or *Affirmative Demonftration*, is that which concludes with the direct and certain proof of the propofition in hand.—This kind of Demonftration is moft fatisfactory to the mind ; for which reafon it is called fometimes an *Oftenfive Demonftration*.

17. *An Indirect* or *Negative Demonftration*, is that which fhews a propofition to be true, by proving that fome abfurdity would neceffarily follow if the propofition advanced were falfe. This is alfo fometimes called *Reductio ad Ab-furdum*; becaufe it fhews the abfurdity and falfehood of all fuppofitions contrary to that contained in the propofition.

18. *Method*, is the art of difpofing a train of arguments in a proper order, to inveftigate either the truth or falfity of a propofition, or to demonftrate it to others when it has been found out.——This is either Analytical or Synthetical.

19. *Analyfis*, or the *Analytic Method*, is the art or mode of finding out the truth of a propofition, by firft fuppofing the thing to be done, and then reafoning back ftep by ftep till we arrive at fome known truth. This is alfo called the *Method of Invention*, or *Refolution*; and is that which is commonly ufed in Algebra.

20. *Synthefis*, or the *Synthetic Method*, is the fearching out truth, by firft laying down fome fimple and eafy principles. and purfuing the confequences flowing from them till we arrive at the conclufion.—This is alfo called the *Method of Compofition*; and is the reverfe of the Analytic method, as this proceeds from known principles to an unknown conclufion ; while the other goes in a retrograde order, from the thing fought, confidered as if it were true, to fome known principle or fact. And therefore, when any truth has been found out by the Analytic method, it may be demonftrated by a procefs in the contrary order, by Synthefis.

ARITH-

# ARITHMETIC.

Arithmetic is the art or fcience of numbering; being that branch of Mathematics which treats of the nature and properties of numbers.—When it treats of whole numbers, it is called *Vulgar*, or *Common Arithmetic;* but when of broken numbers, or parts of numbers, it is called *Fractions*.

*Unity*, or an *Unit*, is that by which every thing is called one; being the beginning of number. As one man, one ball, one gun.

*Number* is either fimply one, or a compound of feveral units. As one man, three men, ten men.

An *Integer*, or *Whole Number*, is fome certain precife quantity of units; as one, three, ten.—Thefe are fo called as diftinguifhed from *Fractions*, which are broken numbers, or parts of numbers; as one-half, two-thirds, or three-fourths.

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## NOTATION and NUMERATION.

Notation, or Numeration, teaches to denote or exprefs any propofed number, either by words or characters; or to read and write down any fum or number.

The numbers in Arithmetic are expreffed by the following ten digits, or Arabic numeral figures, which were introduced into Europe by the Moors, about eight or nine hundred years fince: viz. 1 one, 2 two, 3 three, 4 four, 5 five, 6 fix, 7 feven, 8 eight, 9 nine, o cipher or nothing. Thefe characters or figures were formerly all called by the general name of *Ciphers;* whence it came to pafs, that the art of Arithmetic was then often called *Ciphering.* Alfo the firft nine are called *Significant Figures*, as diftinguifhed from the cipher, which is quite infignificant of itfelf.

Befide this value of thofe figures, they have alfo another, which depends on the place they ftand in when joined together; as in the following Table:

Units

| &c. | Hundreds of Millions | Tens of Millions | Millions | Hundreds of Thousands | Tens of Thousands | Thousands | Hundreds | Tens | Units |
|---|---|---|---|---|---|---|---|---|---|
| &c. | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 |
|  | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 |  |
|  |  | 9 | 8 | 7 | 6 | 5 | 4 | 3 |  |
|  |  |  | 9 | 8 | 7 | 6 | 5 | 4 |  |
|  |  |  |  | 9 | 8 | 7 | 6 | 5 |  |
|  |  |  |  |  | 9 | 8 | 7 | 6 |  |
|  |  |  |  |  |  | 9 | 8 | 7 |  |
|  |  |  |  |  |  |  | 9 | 8 |  |
|  |  |  |  |  |  |  |  | 9 |  |

Here, any figure in the firſt place, reckoning from right to left, denotes only its own ſimple value; but that in the ſecond place, denotes ten times its ſimple value ; and that in the third place, a hundred times its ſimple value ; and ſo on; the value of any figure, in each ſucceſſive place, being always ten times its former value.

Thus, in the number 1796, the 6 in the firſt place denotes only ſix units, or ſimply ſix; 9 in the ſecond place ſignifies nine tens, or ninety ; 7 in the third place, ſeven hundred ; and the 1 in the fourth place, one thouſand ; ſo that the whole number is read thus, one thouſand ſeven hundred and ninety-ſix.

As to the cipher 0, it ſtands for nothing of itſelf, but being joined on the right hand ſide to other figures, it increaſes their value in the ſame tenfold proportion : thus, 5 ſignifies only five; but 50 denotes 5 tens, or fifty ; and 500 is five hundred ; and ſo on.

For the more eaſily reading of large numbers, they are divided into periods and half-periods, each half-period conſiſting of three figures ; the name of the firſt period being units ; of the ſecond, millions ; of the third, millions of millions, or bi-millions, contracted to billions ; of the fourth, millions of millions of millions, or tri-millions, contracted to trillions ; and ſo on. Alſo the firſt part of any period is ſo many units of it, and the latter part ſo many thouſands.

The

The following Table contains a fummary of the whole doctrine:

| Periods. | Quadrill. Trillions; Billions; Millions; Units. |
|---|---|
| Half-per. | th. un. th. un. th. un. th. un. th. un. |
| Figures. | 123,456; 789,098; 765,432; 101,234; 567,890 |

NUMERATION is the reading of any number in words that is propofed or fet down in figures; which will be eafily done by help of the following rule, deduced from the foregoing tablets and obfervations, viz.

Divide the figures in the propofed number, as in the fummary above, into periods and half-periods; then begin at the left-hand fide, and read the figures with the names fet to them in the two foregoing tables.

### EXAMPLES.

Exprefs in words the following numbers; viz.

| | | |
|---|---|---|
| 34 | 15080 | 13405670 |
| 96 | 72003 | 47050023 |
| 180 | 109026 | 309025600 |
| 304 | 483500 | 4723507689 |
| 6134 | 2500639 | 274856390000 |
| 9028 | 7523000 | 6578600307024 |

NOTATION is the fetting down in figures any number propofed in words; which is done by fetting down the figures inftead of the words or names belonging to them in the fummary above; fupplying the vacant places with ciphers where any words do not occur.

### EXAMPLES.

Set down in figures the following numbers:

Fifty-feven.

Two hundred eighty-fix.

Nine thoufand two hundred and ten.

Twenty-feven thoufand five hundred and ninety-four.

Six hundred and forty thoufand, four hundred and eighty-one.

Three millions, two hundred fixty thoufand, one hundred and fix.

Four hundred and eight millions, two hundred and fifty-five thoufand, one hundred and ninety-two.

Twenty-feven thoufand and eight millions, ninety-fix thoufand two hundred and four.

Two hundred thoufand and five hundred and fifty millions, one hundred and ten thoufand, and fixteen.

Twenty-one billions, eight hundred and ten millions, fixty-four thoufand, one hundred and fifty.

### Of the ROMAN NOTATION.

THE Romans, like feveral other nations, expreffed their numbers by certain letters of the alphabet. The Romans ufed only feven numeral letters, being the feven following capitals : viz. I for *one ;* V for *five;* X for *ten ;* L for *fifty;* C for an *hundred;* D for *five hundred;* M for a *thoufand.* The other numbers they expreffed by various repetitions and combinations of thefe, after the following manner :

| | | |
|---|---|---|
| 1 = I | | |
| 2 = II | As often as any character is repeated, fo many times is its value repeated. | |
| 3 = III | | |
| 4 = IIII or IV | A lefs character before a greater diminifhes its value. | |
| 5 = V | | |
| 6 = VI | A lefs character after a greater increafes its value. | |
| 7 = VII | | |
| 8 = VIII | | |
| 9 = IX | | |
| 10 = X | | |
| 50 = L | | |
| 100 = C | | |
| 500 = D or IƆ | For every Ɔ annexed, this becomes 10 times as many. | |
| 1000 = M or CIƆ | For every C and Ɔ, placed one at each end, it becomes 10 times as much. | |
| 2000 = MM | | |
| 5000 = V̄ or IƆƆ | A bar over any number, increafes it 1000 fold. | |
| 6000 = V̄I | | |
| 10000 = X̄ or CCIƆƆ | | |
| 50000 = L̄ or IƆƆƆ | | |
| 60000 = L̄X̄ | | |
| 100000 = C̄ or CCCIƆƆƆ | | |
| 1000000 = M̄ or CCCCIƆƆƆƆ | | |
| 2000000 = M̄M̄ | | |
| &c. | &c. | |

## EXPLANATION OF CERTAIN CHARACTERS.

THERE are various characters or marks, used in Arithmetic, and Algebra, to denote several of the operations and propositions ; the chief of which are as follow :

+ signifies *plus*, or addition.

— - - *minus*, or subtraction.

× or . - multiplication.

÷ - - division.

:: :: .- proportion.

= - - equality.

√ - - square root.

³√ - - cube root, &c.

∽ - - diff. between two numbers when it is not known
which is the greater.

Thus,

5 + 3, denotes that 3 is to be added to 5.

6 — 2, denotes that 2 is to be taken from 6.

7 × 3, or 7 . 3, denotes that 7 is to be multiplied by 3.

8 ÷ 4, denotes that 8 is to be divided by 4.

2 : 3 :: 4 : 6, shews that 2 is to 3 as 4 is to 6.

6 + 4 = 10, shews that the sum of 6 and 4 is equal to 10.

√ 3, or $3^{\frac{1}{2}}$, denotes the square root of the number 3.

³√5, or $5^{\frac{1}{3}}$, denotes the cube root of the number 5.

$7^2$, denotes that the number 7 is to be squared.

$8^3$, denotes that the number 8 is to be cubed.

&c.

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## OF ADDITION.

ADDITION is the collecting or putting of several numbers together, in order to find their *sum*, or the total amount of the whole. This is done as follows :

Set or place the numbers under each other, so that each figure may stand exactly under the figures of the same value,

that

that is, units under units, tens under tens, hundreds under hundreds, &c. ; and draw a line under the lowest number, to separate the given numbers from their sum, when it is found. —Then add up the figures in the column or row of units, and find how many tens are contained in their sum.—Set down exactly below, what remains more than those tens, or if nothing remains, a cipher, and carry as many ones to the next row as there are tens.—Next add up the second row, together with the number carried, in the same manner as the first. And thus proceed till the whole is finished, setting down the total amount of the last row.

## To prove Addition.

*First Method.*—Begin at the top, and add together all the rows of numbers downwards ; in the same manner as they were before added upwards ; then if the two sums agree, it may be presumed the work is right.—This method of proof is only doing the same work twice over, a little varied.

*Second Method.*—Draw a line below the uppermost number, and suppose it cut off.—Then add all the rest of the numbers together in the usual way, and set their sum under the number that is to be proved.—Lastly, add this last found number and the uppermost line together ; then if their sum be the same as that found by the first addition, it may be presumed the work is right.—This method of proof is founded on the plain axiom, that " The whole is equal to all its parts taken together."

*Third Method.*—Add the figures in the uppermost line together, and find how many nines are contained in their sum.—Reject those nines, and set down the remainder towards the right hand directly even with the figures in the line, as in the annexed example.—Do the same with each of the proposed lines of numbers, setting all these excesses of nines in a column on the right-hand, as here 5, 5, 6. Then, if the excess of 9's in this sum, found as before, be equal to the excess of 9's in the total sum 18304, the work is right.— Thus, the sum of the right-hand column 5, 5, 6, is 16, the excess of which above 9 is 7. Also the sum of the figures in the

**EXAMPLE I.**

| | Excess of nines | |
|---|---|---|
| 3497 | | 5 |
| 6512 | | 5 |
| 8295 | | 6 |
| 18304 | | 7 |

the

the fum total 18304 is 16, the excefs of which above 9 is alfo 7, the fame as the former*.

OTHER EXAMPLES.

| 2. | 3. | 4. |
|---|---|---|
| 12345 | 12345 | 12345 |
| 67890 | 67890 | 876 |
| 98765 | 9876 | 9087 |
| 43210 | 543 | 56 |
| 12345 | 21 | 234 |
| 67890 | 9 | 1012 |
| 302445 | 90684 | 23610 |
| 290100 | 78339 | 11265 |
| 302445 | 90684 | 23610 |

* This method of proof depends on a property of the number 9, which, except the number 3, belongs to no other digit whatever; namely, that "any number divided by 9, will leave the fame remainder as the fum of its figures or digits divided by 9;" which may be demonftrated in this manner.

*Demonftration.* Let there be any number propofed, as 4658. This, feparated into its feveral parts, becomes 4000 + 600 + 50 + 8. But 4000 = 4 × 1000 = 4 × (999 + 1) = 4 × 999 + 4. In like manner 600 = 6 × 99 + 6; and 50 = 5 × 9 + 5. Therefore the given number 4658 = 4 × 999 + 4 + 6 × 99 + 6 + 5 × 9 + 5 + 8 = 4 × 999 + 6 × 99 + 5 × 9 + 4 + 6 + 5 + 8; and 4658 ÷ 9 = (4 × 999 + 6 × 99 + 5 × 9 + 4 + 6 + 5 + 8) ÷ 9. But 4 × 999 + 6 × 99 + 5 × 9 is evidently divifible by 9, without a remainder; therefore if the given number 4658 be divided by 9, it will leave the fame remainder as 4 + 6 + 5 + 8 divided by 9. And the fame, it is evident, will hold for any other number whatever.

In like manner, the fame property may be fhewn to belong to the number 3; but the preference is ufually given to the number 9, on account of its being more convenient in practice.

Now, from the demonftration above given, the reafon of the rule itfelf is evident; for the excefs of 9's in two or more numbers being taken feparately, and the excefs of 9's taken alfo out of the fum of the former exceffes, it is plain that this laft excefs muft be equal to the excefs of 9's contained in the total fum of all thefe numbers; all the parts taken together being equal to the whole.
——This rule was firft given by Dr. Wallis in his Arithmetic, publifhed in the year 1657.

Ex.

Ex. 5. Add 3426; 9024; 5106; 8390; 1204 together.
                                          Anf. 27150.

6. Add 509267; 235809: 72910; 8392; 420; 21; and 9 together.                                Anf. 826828.

7. Add 2; 19; 817; 4298; 50916; 730205; 9120634 together.                              Anf. 9906891.

8. How many days are in the twelve calendar months?
                                          Anf. 365.

9. How many days are there from the 15th day of April to the 24th day of November, both days included?   Anf. 224.

10. An army confifting of 52714 infantry * or foot, 5110 horfe, 6250 dragoons, 3927 light-horfe, 928 artillery or gunners, 1410 pioneers, 250 fappers, and 406 miners; what is the whole number of men?                   Anf. 70995.

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## OF SUBTRACTION.

SUBTRACTION teaches to find how much one number exceeds another, called their *difference*, or the *remainder*, by taking the lefs from the greater. The method of doing which is as follows:

Place the lefs number under the greater, in the fame manner as in Addition, that is, units under units, tens under tens, and fo on; and draw a line below them.—Begin at the right-hand, and take each figure in the lower line or number from the figure above it, fetting down the remainder below it.— But if the figure in the lower line be greater than that above it, firft borrow or add 10 to the upper one, and then take the lower figure from that fum, fetting down the remainder, and carrying 1, for what was borrowed, to the next lower figure, with which proceed as before, and fo on till the whole is finifhed.

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* The whole body of foot foldiers is denoted by the word *Infantry*; and all thofe that charge on horfeback, by the word *Cavalry*.—Some authors conjecture, that the term infantry is derived from a certain Infanta of Spain, who, finding that the army commanded by the king her father had been defeated by the Moors, affembled a body of the people together on foot, with which fhe engaged and totally routed the enemy. In honour of this event, and to diftinguifh the foot foldiers, who were not before held in much eftimation, they received the name of Infantry, from her own title of Infanta.

To

## To prove Subtraction.

Add the remainder to the lefs number, or that which is juft above it, and if the fum be equal to the greater or uppermoft number, the work is right *.

### EXAMPLES.

|  | 1. |  | 2. |  | 3. |
|---|---|---|---|---|---|
| From | 5386427 | From | 5386427 | From | 1234567 |
| Take | 2164315 | Take | 4258792 | Take | 702973 |
| Rem. | 3222112 | Rem. | 1127635 | Rem. | 531594 |
| Proof. | 5386427 | Proof. | 5386427 | Proof. | 1234567 |

4. From 5231806   Take 5073918.     Anf.   157888.
5. From 7020914   Take 2766809.     Anf. 4254105.
6. From 8503602   Take   574371.     Anf. 7929231.

7. Sir Ifaac Newton was born in the year 1642, and he died in 1727 ; how old was he at the time of his deceafe ?
                        Anf. 85 years.

8. Homer was born 2530 years ago, and Chrift 1797 years ago ; then how long before Chrift was the birth of Homer ?                       Anf. 733 years.

9. Noah's flood happened about the year of the world 1656, and the birth of Chrift about the year 4000 ; then how long was the flood before Chrift ?         Anf. 2344 years.

10. The Arabian or Indian method of notation was firft known in England about the year 1150; then how long is it fince to this prefent year 1797 ?       Anf. 647 years.

11. Gunpowder was invented in the year 1330 ; then how long was this before the invention of printing, which was in 1441 ?                      Anf. 111 years.

12. The mariner's compafs was invented in Europe in the year 1302 ; then how long was that before the difcovery of America by Columbus, which happened in 1492 ?
                        Anf. 190 years.

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* The reafon of this method of proof is evident : for if the difference of two numbers be added to the lefs, it muft manifeftly make up a fum equal to the greater.

## OF MULTIPLICATION.

MULTIPLICATION is a compendious method of Addition, teaching how to find the amount of any given number when repeated a certain number of times. As 4 times 6, which is 24.

The number to be multiplied, or repeated, is called the *Multiplicand.*—The number you multiply by, or number of repetitions, is the *Multiplier.*—And the number found, being the total amount, is called the *Product.*—Also, both the multiplier and multiplicand are, in general, named the *Terms* or *Factors.*

Before proceeding to any operations in this rule it is neceſſary to learn off very perfectly the following Table of all the products of the firſt 12 numbers, sometimes called the Multiplication Table, or Pythagoras's Table, from its inventor.

### MULTIPLICATION TABLE.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 |
| 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 | 30 | 33 | 36 |
| 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 | 40 | 44 | 48 |
| 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 | 55 | 60 |
| 6 | 12 | 18 | 24 | 30 | 36 | 42 | 48 | 54 | 60 | 66 | 72 |
| 7 | 14 | 21 | 28 | 35 | 42 | 49 | 56 | 63 | 70 | 77 | 84 |
| 8 | 16 | 24 | 32 | 40 | 48 | 56 | 64 | 72 | 80 | 88 | 96 |
| 9 | 18 | 27 | 36 | 45 | 54 | 63 | 72 | 81 | 90 | 99 | 108 |
| 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 |
| 11 | 22 | 33 | 44 | 55 | 66 | 77 | 88 | 99 | 110 | 121 | 132 |
| 12 | 24 | 36 | 48 | 60 | 72 | 84 | 96 | 108 | 120 | 132 | 144 |

*To multiply any Given Number by a Single Figure, or by any Number not more than 12.*

\* Set the multiplier under the units figure, or right-hand place, of the multiplicand, and draw a line below it.—Then, beginning at the right hand, multiply every figure in this by the multiplier.—Count how many tens there are in the product of every single figure, and set down the remainder directly under the figure that is multiplied; and if nothing remains, set down a cipher.—Carry as many units or ones, as there are tens counted, to the product of the next figures; and proceed in the same manner till the whole is finished.

EXAMPLE.

Multiply 9876543210    Multiplicand.
By     -   -   2    Multiplier.

   ─────────────

   19753086420

   ─────────────

*To multiply by a Number consisting of Several Figures.*

† Set the multiplier below the multiplicand, placing them as in Addition, namely units under units, tens under tens, &c. drawing a line below it.—Multiply the whole of the multiplicand by each figure of the multiplier, as in the last article;

                                                    setting

─────────────────────────────────────────────

\* The reason of this rule is the same as for the process in Addition, in which 1 is carried for every 10, to the next place, gradually as the several products are produced, one after another, instead of setting them all down below each other, as in the annexed Example.

$$5678$$
$$4$$
─────
$$32 = 8 \times 4$$
$$280 = 70 \times 4$$
$$2400 = 600 \times 4$$
$$20000 = 5000 \times 4$$
─────────────
$$22712 = 5678 \times 4$$

† After having found the produce of the multiplicand by the first figure of the multiplier, as in the former case, the multiplier is supposed to be divided into parts, and the product is found for the second figure in the same manner: but as this figure stands in the place of tens, the product must be ten times its simple value; and therefore the first figure of this product must be set in the place of tens; or,

                                                    which

fetting down a line of products for each figure in the multiplier, fo as that the firft figure of each line may ftand ftraight under the figure multiplying by.—Add all the lines of products together, in the order as they ftand, and their fum will be the anfwer or whole product required.

## To prove Multiplication.

There are three different ways of proving Multiplication, which are as below :

*Firft Method.*—Make the multiplicand and multiplier change places, and multiply the latter by the former in the fame manner as before. Then if the product found in this way be the fame as the former, the number is right.

*Second Method.*—*Caft all the 9's out of the fum of the figures in each of the two factors, as in Addition, and fet down the remainders. Multiply thefe two remainders together, and caft the 9's out of the product, as alfo out of

———————————————————————————

which is the fame thing, directly under the figure multiplied by. And proceeding in this manner feparately with all the figures of the multiplier, it is evident that we fhall multiply all the parts of the multiplicand by all the parts of the multiplier, or the whole of the multiplicand : therefore thefe feveral products being added together, will be equal to the whole required product ; as in the example annexed.

| | |
|---|---|
| 1234567 | the multiplicand. |
| 4567 | |
| ——————— | |
| 8641969 = | 7 times the mult. |
| 7407402 = | 60 times ditto. |
| 6172835 = | 500 times ditto. |
| 4938268 | =4000 times ditto. |
| ——————— —— | —— |
| 5638267489=4567 times ditto. |

* This method of proof is derived from the peculiar property of the number 9, mentioned in the proof of Addition, and the reafon for the one may ferve for that of the other. Another more ample demonftration of this rule may be as follows :—Let P and Q denote the number of 9's in the factors to be multiplied, and $a$ and $b$ what remain ; then $9P + a$ and $9Q + b$ will be the numbers themfelves, and their product is $(9P \times 9Q) + (9P \times b) + (9Q \times a) + (a \times b)$ ; but the firft three of thefe products are each a precife number of 9's, becaufe their factors are fo, either one or both : thefe therefore being caft away, there remains only $a \times b$ ; and if the 9's be alfo caft out of this, the excefs is the excefs of 9's in the total product : but $a$ and $b$ are the excefles in the factors themfelves, and $a \times b$ is their product ; therefore the rule is true.

the

the whole product or anfwer of the queftion, referving the remainders of thefe laft two, which remainders muft be equal when the work is right.—*Note*, It is common to fet the four remainders within the four angular fpaces of a crofs, as in the example below.

*Third Method.*—Multiplication is alfo very naturally proved by Divifion; for the product divided by either of the factors, will evidently give the other. But this cannot be practifed till the rule of Divifion is learned.
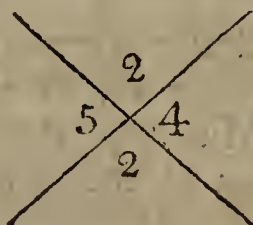
### EXAMPLES.

Mult. 3542
by 6196
———
21252
31878
3542
21252
———
21946232 Product

Proof.
2
5 X 4
2

or Mult. 6196
by 3542
———
12392
24784
30980
18588
———
21946232 Proof.

### OTHER EXAMPLES.

Multiply 123456789 by 3.    Anf. 370370367.
Multiply 123456789 by 4.    Anf. 493827156.
Multiply 123456789 by 5.    Anf. 617283945.
Multiply 123456789 by 6.    Anf. 740740734.
Multiply 123456789 by 7.    Anf. 864197523.
Multiply 123456789 by 8.    Anf. 987654312.
Multiply 123456789 by 9.    Anf. 1111111101.
Multiply 123456789 by 11.    Anf. 1358024679.
Multiply 123456789 by 12.    Anf. 1481481468.
Multiply 302914603 by 16.    Anf. 4846633648.
Multiply 273580961 by 23.    Anf. 6292362103.
Multiply 402097316 by 195.    Anf. 78408976620.
Multiply 82164973 by 3027.    Anf. 248713373271.
Multiply 7564900 by 579.    Anf. 4380077100.
Multiply 8496427 by 874359. Anf. 7428927415293.
Multiply 2760325 by 37072.  Anf. 102330768400.

## CONTRACTIONS IN MULTIPLICATION.

### I. *When there are Ciphers in the Factors.*

IF the ciphers be at the right-hand of the numbers; multiply the other figures only, and annex as many ciphers to the right-hand of the product, as are in both the factors.—And when the ciphers are in the middle parts of the multiplier; neglect them as before, only taking care to place the first figure of every line of products exactly under the figure multiplying with.

### EXAMPLES.

|  1.  |  2.  |
|---|---|
| Mult. 9001635 | Mult. 390720400 |
| by -  70100 | by -  406000 |

```
      9001635              23443224
   63011445              15628816

  631014613500 Products 158632482400000
```

3. Multiply 81503600 by 7030.  Anf. 572970308000.
4. Multiply 9030100  by 2100.  Anf. 18963210000.
5. Multiply 8057069  by 70050.  Anf. 564397683450.

### II. *When the Multiplier is the Product of two or more Numbers in the Table; then*

* Multiply by each of those parts separately, instead of the whole number at once.

### EXAMPLES.

1. Multiply 51307298 by 56, or 7 times 8.

```
        51307298
               7

       359151086
               8

      2873208688
```

---

* The reason of this rule is obvious enough; for any number multiplied by the component parts of another, must give the same product as if it were multiplied by that number at once. Thus, in the 1st example, 7 times the product of 8 by the given number, makes 56 times the same number, as plainly as 7 times 8 makes 56.

2. Multiply 31704592   by 36.     Anf. 1141365312.
3. Multiply 29753804   by 72.     Anf. 2142273888.
4. Multiply 7128368     by 96.     Anf. 684323328.
5. Multiply 160430800 by 108.     Anf. 17326526400.
6. Multiply 61835720   by 1320.    Anf. 81623150400.

7. There was an army compofed of 104 *battalions, each confifting of 500 men ; what was the number of men contained in the whole ?           Anf. 52000.

8. A convoy of ammunition † bread, confifting of 250 waggons, and each waggon containing 320 loaves, having been intercepted and taken by the enemy ; what is the number of loaves loft ?           Anf. 80000.

## OF DIVISION.

DIVISION is a kind of compendious method of Subtraction, teaching to find how often one number is contained in another, or may be taken out of it, which is the fame thing.

The number to be divided is called the *Dividend*.— The number to divide by, is the *Divifor*.—And the number of times the dividend contains the divifor, is called the *Quotient*.—Sometimes there is a *Remainder* left, after the divifion is finifhed.

The ufual manner of placing the terms, is, the dividend in the middle, having the divifor on the left-hand, and the quotient on the right, each feparated by a curve line ; as, to divide 12 by 4, the quotient is 3,

| Dividend | | 12 |
|---|---|---|
| Divifor 4)    12    ( 3 Quotient ; | | 4 fubtr. |
| fhewing that the number 4 is 3 times | | — |
| contained in 12, or may be three times | | 8 |
| fubtracted out of it, as in the margin. | | 4 fubtr. |
| ‡ *Rule.*—Having placed the divifor | | — |
| before the dividend, as above direct- | | 4 |
| ed, find how often the divifor is con- | | 4 fubtr. |
| tained in as many figures of the divi- | | — |
| dend as are juft neceffary, and place the | | 0 |
| number on the right in the quotient. | | — |

Mul-

---

* A battalion is a body of foot, confifting of 500, or 600, or 700 men, more or lefs.

† The ammunition bread is that which is provided for, and diftributed to the foldiers ; the ufual allowance being a loaf of 6 pounds to every foldier, once in 4 days.

‡ In this way we refolve the dividend into parts, and find by trial how

Multiply the divisor by this number, and set the product under the figures of the dividend before-mentioned.—Subtract this product from that part of the dividend under which it stands, and bring down the next figure of the dividend, or more if neceſſary, to join on the right of the remainder.— Divide this number, ſo increaſed, in the ſame manner as before; and ſo on till all the figures are brought down and uſed.

N. B. If it be neceſſary to bring down more figures than one to any remainder, in order to make it as large as the divisor, or larger, a cipher muſt be ſet in the quotient for every figure ſo brought down more than one.

## To prove Division.

\* Multiply the quotient by the divisor ; to this product add the remainder, if there be any; then the ſum will be equal to the dividend when the work is right.

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how often the divisor is contained in each of thoſe parts, one after another, and arranging the ſeveral figures of the quotient one after another, into one number.

When there is no remainder to a diviſion, the quotient is the whole and perfect anſwer to the queſtion. But when there is a remainder, it goes ſo much towards another time as it approaches to the divisor : ſo, if the remainder be half the divisor, it will go the half of a time more; if the 4th part of the divisor, it will go one fourth of a time more ; and ſo on. Therefore, to complete the quotient, ſet the remainder at the end of it, above a ſmall line, and the divisor below it, thus forming a fractional part of the whole quotient.

. \* This method of proof is plain enough : for ſince the quotient is the number of times the dividend contains the divisor, the quotient multiplied by the divisor muſt evidently be equal to the dividend.

There are alſo ſeveral other methods ſometimes uſed for proving Diviſion, ſome of the moſt uſeful of which are as follow :

*Second Method.*—Subtract the remainder from the dividend, and divide what is left by the quotient ; ſo ſhall the new quotient from this laſt diviſion be equal to the former divisor, when the work is right.

*Third Method.*—Add together the remainder and all the products of the ſeveral quotient figures by the divisor, according to the order in which they ſtand in the work ; and the ſum will be equal to the dividend when the work is right.

## EXAMPLES.

| 1. | Quot. | | 2. | Quot. |
|---|---|---|---|---|

```
  3) 1234567 ( 411522          37 ) 12345678 ( 333666
     12        mult. 3              111                37
    ────       ─────             ─────            ───────
      3         1234566           124             2335662
      3        add 1              111             1000998
    ────       ─────             ─────          rem. 36
      4         1234567           135           ───────
      3        ───────           111            12345678
    ────        Proof.          ─────          ───────
     15                          246             Proof.
     15                          222
    ────                       ─────
      6                          247
      6                          222
    ────                       ─────
      7                          258
      6                          222
    ────                       ─────
    Rem. 1                      Rem. 36
```

3. Divide 73146085    by 4.        Anf. $18286521\frac{1}{4}$.

4. Divide 5317986027 by 7.        Anf. $759712289\frac{4}{7}$.

5. Divide 570196382 by 12.        Anf. $47516365\frac{2}{12}$.

6. Divide 74638105    by 37.        Anf. $2017246\frac{3}{37}$.

7. Divide 137896254 by 97.        Anf. $1421610\frac{84}{97}$.

8. Divide 35821649    by 764.       Anf. $46886\frac{745}{764}$.

9. Divide 72091365    by 5201.      Anf. $13861\frac{304}{5201}$.

10. Divide 463706283 by 57606.    Anf. $80496\frac{11707}{57606}$.

11. Suppofe 471 men are formed into ranks of 3 deep, what is the number in each rank ?        Anf. 157.

12. A party, at the diftance of 378 miles from the head quarters, receive orders to join their corps in 18 days ; what number of miles muft they march each day to obey their orders ?        Anf. 21.

13. The annual revenue of a gentleman being 38330l ; how much per day is that equivalent to, there being 365 days in the year ?        Anf. 104l.

### CONTRACTIONS IN DIVISION.

THERE are certain contractions in Divifion, by which the operation in particular cafes may be performed in a fhorter manner ; as follows :

                     I. *Divi-*

I. *Division by any Small Number*, not greater than 12, may be expeditiously performed, by multiplying and subtracting mentally, omitting to set down the work, except only the quotient immediately below the dividend.

<div align="center">EXAMPLES.</div>

3 ) 56103961      4 ) 52019675      5 )  1370192

Quot.  18701320⅓

6 ) 38072940      7 ) 81390627      8 ) 23718620

9 ) 43081962      11 ) 57014230      12 ) 27980313

II. * *When Ciphers are annexed to the Divisor ;* cut off those ciphers from it, and cut off the same number of figures from the right-hand of the dividend ; then divide the remaining figures, as usual. And if there be any thing remaining after this division, place the figures cut off from the dividend to the right of it, and the whole will be the true remainder ; otherwise, the figures cut off only will be the remainder.

<div align="center">EXAMPLES.</div>

1. Divide 3704196 by 20.      2. Divide 31086901 by 7100.

2,0 ) 370419,6            71,00 ) 310869,01  ( 4378$\frac{3101}{7100}$..
                                284
Quot.  185209$\frac{16}{20}$
                                  268
                                  213

                                    556
                                    497

                                    599
                                    568

                                      31

3. Divide

---

3. Divide 7380964 by 23000.          Anf. 320$\frac{20964}{23000}$.

4. Divide 2304109 by 5800.          Anf. 397$\frac{1509}{5800}$.

III. *When the Divifor is the exaƐt ProduƐt of two or more of the fmall Numbers not greater than* 12: \* Divide by each of thofe numbers feparately, inftead of the whole divifor at once.

·*N. B.* There are commonly feveral remainders in working by this rule, one to each divifion ; and to find the true or whole remainder, the fame as if the divifion had been performed all at once, proceed as follows : Multiply the laft remainder by the preceding divifor, or laft but one, and to the produƐt add the preceding remainder ; multiply this fum by the next preceding divifor, and to the produƐt add the next preceding remainder ; and fo on, till you have gone backward through all the divifors and remainders to the firft. As in the example following :

### EXAMPLES.

1. Divide 31046835 by 56 or 7 times 8.

7 ) 31046835                     6 the laft rem.

8 )    4435262——1 firft rem.    mult. 7 preced. divifor.

                                           ——

                                           42

554407——6 fecond rem.  add 1 the 1ft. rem.

                                           ——

Anf. 554407$\frac{43}{56}$.                     43 whole rem.

---

principle upon which it is founded, is evident ; for, cutting off the fame number of ciphers, or figures, from each, is the fame as dividing each of them by 10, or 100, or 1000, &c. according to the number of ciphers cut off ; and it is evident that as often as the whole divifor is contained in the whole dividend, fo often muft any part of the former be contained in a like part of the latter.

\* This follows from the 2d contraƐion in Multiplication, being only the converfe of it ; for the half of the third part of any thing, is evidently the fame as the fixth part of the whole ; and fo of any other numbers.——The reafon of the method of finding the whole remainder from the feveral particular ones, will beft appear from the nature of Vulgar FraƐions. Thus, in the firft example above, the firft remainder being 1, when the divifor is 7, makes $\frac{1}{7}$ ; this muft be added to the fecond remainder 6, making 6$\frac{1}{7}$ to the divifor 8, or to be divided by 8. But 6$\frac{1}{7}$ =

$\frac{6 \times 7 + 1}{7} = \frac{43}{7}$; and this divided by 8 gives $\frac{43}{7 \times 8} = \frac{43}{56}$.

2. Divide

2. Divide 7014596 by 72.      Anf. 97424$\frac{68}{72}$.
3. Divide 5130652 by 132.      Anf. 38868$\frac{76}{132}$.
4. Divide 83016572 by 240.      Anf. 345902$\frac{92}{240}$.

IV. *Common Division may be performed more concisely,* by omitting the several products, and setting down only the re-mainders; namely, multiply the divisor by the quotient figures as before, and, without setting down the product, subtract each figure of it from the dividend, as it is produced; always remembering to carry as many to the next figure as were borrowed before.

<div align="center">E X A M P L E S.</div>

<div align="center">1. Divide 304679 by 833.</div>

$$833 ) 3104679 ( 3727\tfrac{88}{833}.$$
$$6056$$
$$2257$$
$$5919$$
$$88$$

2. Divide 79165238 by 238.      Anf. 332627$\frac{12}{238}$.
3. Divide 29137062 by 5317.      Anf. 5479$\frac{5219}{5317}$.
4. Divide 62015735 by 7803.      Anf. 7947$\frac{5294}{7803}$.

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# OF REDUCTION.

REDUCTION is the changing of numbers from one name or denomination to another, without altering their value.—This is chiefly concerned in reducing money, weights, and measures.

When the numbers are to be reduced from a higher name to a lower, it is called *Reduction Descending;* but when, con-trarywise, from a lower name to a higher, it is *Reduction Ascending*.

Before proceeding to the rules and questions of Reduction, it will be proper to set down the usual Tables of money, weights, and measures, which are as follow:

## Of MONEY, WEIGHTS, AND MEASURES.

### TABLES OF MONEY.*

| | | | qrs | | d | |
|---|---|---|---|---|---|---|
| 2 Farthings | = 1 Halfpenny $\frac{1}{2}$ | | | | | |
| 4 Farthings | = 1 Penny | d | 4 = | 1 | | s |
| 12 Pence | = 1 Shilling | s | 48 = | 12 = | 1 | £ |
| 20 Shillings | = 1 Pound | £ | 960 = | 240 = | 20 = | 1 |

| PENCE TABLE. | | | | | SHILLINGS TABLE. | | |
|---|---|---|---|---|---|---|---|
| d. | | s. | d. | | s. | | d. |
| 20 | is | 1 | 8 | | 1 | is | 12 |
| 30 | — | 2 | 6 | | 2 | — | 24 |
| 40 | — | 3 | 4 | | 3 | — | 36 |
| 50 | — | 4 | 2 | | 4 | — | 48 |
| 60 | — | 5 | 0 | | 5 | — | 60 |
| 70 | — | 5 | 10 | | 6 | — | 72 |
| 80 | — | 6 | 8 | | 7 | — | 84 |
| 90 | — | 7 | 6 | | 8 | — | 96 |
| 100 | — | 8 | 4 | | 9 | — | 108 |
| 110 | — | 9 | 2 | | 10 | — | 120 |
| 120 | — | 10 | 0 | | 11 | — | 132 |

TROY

* £ denotes pounds, s shillings, and d denotes pence.

$\frac{1}{4}$ denotes 1 farthing, or one quarter of any thing.

$\frac{1}{2}$ denotes a halfpenny, or the half of any thing.

$\frac{3}{4}$ denotes 3 farthings, or 3 quarters of any thing.

The full weight and value of the English gold and silver coin, is as here below :

| GOLD. | Value. | | Weight. | | SILVER. | Value. | | Weight. | |
|---|---|---|---|---|---|---|---|---|---|
| | £ s d | | dwt. gr. | | | s d | | dwt. gr. | |
| A Guinea | 1 1 0 | | 5 | $9\frac{1}{2}$ | A Crown | 5 0 | | 19 | $8\frac{1}{2}$ |
| Half-guinea | 0 10 6 | | 2 | $16\frac{3}{4}$ | Half-crown | 2 6 | | 9 | $16\frac{1}{4}$ |
| Seven Shillings | 0 7 0 | | 1 | $19\frac{1}{6}$ | Shilling | 1 0 | | 3 | 21 |
| Quarter guinea | 0 5 3 | | 1 | $8\frac{1}{4}$ | Sixpence | 0 6 | | 1 | $22\frac{1}{2}$ |

The usual value of gold is nearly 4l an ounce, or 2d a grain; and that of silver is nearly 5s an ounce. Also, the value of any quantity of gold, is to the value of the same weight of standard silver, nearly as 15 to 1, or more nearly as 15 and 1-14th to 1.

Pure gold, free from mixture with other metals, usually called fine gold, is of such purity of nature, that it will endure the fire without

## TROY WEIGHT.*

| Grains | - | - | - | marked *gr* | *gr* | | | *dwt* | |
|---|---|---|---|---|---|---|---|---|---|
| 24 Grains make 1 Pennyweight *dwt* | | | | | 24 | = | 1 | | *oz* |
| 20 Pennyweights 1 Ounce | | | | *oz* | 480 | = | 20 | = 1 | *lb* |
| 12 Ounces | | 1 Pound | | *lb* | 5760 | = | 240 | =12=1 | |

By this weight are weighed Gold, Silver, and Jewels.

## APOTHECARIES' WEIGHT.

| Grains | - | - | | marked | *gr* | |
|---|---|---|---|---|---|---|
| 20 Grains make | 1 Scruple | - - - | *fc* | or | ℈ | |
| 3 Scruples | - - 1 Dram | - - - | *dr* | or | ℥ | |
| 8 Drams | - - 1 Ounce | - - - | *oz* | or | ℥ | |
| 12 Ounces | - - 1 Pound | - - - | *lb* | or | lb | |

| *gr* | | *fc* | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 20 | = | 1 | | *dr* | | | | |
| 60 | = | 3 | = | 1 | | *oz* | | |
| 480 | = | 24 | = | 8 | = | 1 | | *lb* |
| 5760 | = | 288 | = | 96 | = | 12 | = | 1 |

This is the fame as Troy weight, only having fome different divifions. Apothecaries make ufe of this weight in compounding their Medicines; but they buy and fell their Drugs by Avoirdupois weight.

AVOIR-

___

without wafting, though it be kept continually melted. But filver, not having the purity of gold, will not endure the fire like it: yet fine filver will wafte but a very little by being in the fire any moderate time; whereas copper, tin, lead, &c. will not only wafte, but may be calcined or burnt to a powder.

Both gold and filver, in their purity, are fo very foft and flexible (like new lead, &c.) that they are not fo ufeful either in coin or otherwife (except to beat into leaf gold or filver) as when they are allayed, or mixed and hardened with copper or brafs. And though moft nations differ, more or lefs, in the quantity of fuch allay, as well as in the fame place at different times, yet in England the ftandard for gold and filver coin has been for a long time as follows, viz. That 22 parts of fine gold, and 2 parts of copper, being melted together, fhall be efteemed the true ftandard for gold coin: And that 11 ounces and 2 pennyweights of fine filver, and 18 pennyweights of copper, being melted together, is efteemed the true ftandard for filver coin, called Sterling filver.

* The original of all weights ufed in England, was a grain or corn of wheat, gathered out of the middle of the ear, and, being well dried, 32 of them were to make one pennyweight, 20 pennyweights

one

## AVOIRDUPOIS WEIGHT.

Drams            -       -       -       marked   *dr*
16 Drams   -   make   1 Ounce  -   -   -   -   *oz*
16 Ounces   -   -   -   1 Pound   -   -   -   -  *lb*
28 Pounds   -   -   -   1 Quarter   -   -   -   *qr*
 4 Quarters   -   -   1 Hundred Weight   *cwt*
20 Hundred Weight   1 Ton   -   -   -   -   *ton*

| Dr | oz | | | | |
|----|----|----|----|----|----|
| 16 = | 1 | *lb* | | | |
| 256 = | 16 = | 1 | *qr* | | |
| 7168 = | 448 = | 28 = | 1 | *cwt* | |
| 28672 = | 1792 = | 112 = | 4 = | 1 | *ton* |
| 573440 = | 35840 = | 2240 = | 80 = | 20 = | 1 |

By this Weight are weighed all things of a coarse or drossy nature, as Corn, Bread, Butter, Cheese, Flesh, Grocery Wares, and some Liquids : also all Metals except Silver and Gold.

*Note*, that 1*lb* Avoirdupois $=$ oz dwt gr 14 11 15½ Troy.

    1*oz*   -   -   $=$   0 18 5½
    1*dr*   -   -   $=$   0 1 3½

## LONG MEASURE.

 3 Barley-corns make   1 Inch   -   -   *In*
12 Inches   -   -   1 Foot   -   -   *Ft*
 3 Feet   -   -   1 Yard   -   -   *Yd*
 6 Feet   -   -   1 Fathom   -   *Fth*
 5 Yards and a half   1 Pole or Rod   -   *Pl*
40 Poles   -   -   1 Furlong   -   *Fur*
 8 Furlongs   -   1 Mile   -   -   *Mile*
 3 Miles   -   -   1 League   -   -   *Lea*
69⅙ Miles nearly   -   1 Degree   -   -   *Deg.* or °

---

one ounce, and 12 ounces one pound. But in later times, it was thought sufficient to divide the same pennyweight into 24 equal parts, still called grains, being the least weight now in common use; and from thence the rest are computed, as in the Tables above.

*In*

| In | Ft | | | | |
|---|---|---|---|---|---|
| 12 = | 1 | *Yd* | | | |
| 36 = | 3 = | 1 | *Pl* | | |
| 198 = | 16½ = | 5½ = | 1 | *Fur* | |
| 7920 = | 660 = | 220 = | 40 = | 1 | *Mile* |
| 63360 = | 5280 = | 1760 = | 320 = | 8 = | 1 |

### Cloth Measure.

| | | | | |
|---|---|---|---|---|
| 2 Inches and a quarter make | 1 Nail | - | - | *Nl* |
| 4 Nails | - | - | 1 Quarter of Yard | *Qr* |
| 3 Quarters - | - | - | 1 Ell Flemish - | *E F* |
| 4 Quarters - | - | - | 1 Yard - | *Yd* |
| 5 Quarters - | - | - | 1 Ell English - | *E E* |
| 4 Qrs 1⅕ Inc | - | - | 1 Ell Scotch - | *E S* |

### Square Measure.

| | | | |
|---|---|---|---|
| 144 | Square Inches make | 1 Sq Foot - | *Ft* |
| 9 | Square Feet - - | 1 Sq Yard - | *Yd* |
| 30¼ | Square Yards - | 1 Sq Pole - | *Pole* |
| 40 | Square Poles - | 1 Rood - | *Rd* |
| 4 | Roods - - | 1 Acre - | *Acr* |

| Sq Inc | Sq Ft | | | | |
|---|---|---|---|---|---|
| 144 = | 1 | *Sq Yd* | | | |
| 1296 = | 9 = | 1 | *Sq Pl* | | |
| 39204 = | 272¼ = | 30¼ = | 1 | *Rd* | |
| 1568160 = | 10890 = | 1210 = | 40 = | 1 | *Acr* |
| 6272640 = | 43560 = | 4840 = | 160 = | 4 = | 1 |

By this measure, Land, Husbandmen and Gardeners work
are measured; also Artificers works, such as Board, Glass,
Pavements, Plastering, Wainscotting, Tiling, Flooring, and
every dimension of length and breadth only.

When three dimensions are concerned, namely, length,
breadth, and depth or thickness, it is called cubic or solid
measure, which is used to measure Timber, Stone, &c.

The cubic or solid Foot, which is 12 inches in length and
breadth and thickness, contains 1728 cubic or solid inches,
and 27 solid feet make one solid yard.

DRY

## DRY, or CORN MEASURE.

| | | | | |
|---|---|---|---|---|
| 2 Pints make | 1 Quart | - | - | Qt |
| 2 Quarts | - 1 Pottle | - | - | Pot |
| 2 Pottles | - 1 Gallon | - | - | Gal |
| 2 Gallons | - 1 Peck | - | - | Pec |
| 4 Pecks | - 1 Bushel | - | - | Bu |
| 8 Bushels | - 1 Quarter | - | - | Qr |
| 5 Quarters | - 1 Weigh or Load | | | Wey or Ton |
| 2 Weys | - 1 Last | - | - | Last |

| Pts | | Gal | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 8 = | | 1 = | Pec | | | | | |
| 16 = | | 2 = | 1 = | Bu | | | | |
| 64 = | | 8 = | 4 = | 1 | Qr | | | |
| 512 = | | 64 = | 32 = | 8 = | 1 | Wey | | |
| 2560 = | | 320 = | 160 = | 40 = | 5 = | 1 | Last | |
| 5120 = | | 640 = | 320 = | 80 = | 10 = | 2 = | 1 | |

By this are meafured all dry wares, as Corn, Seeds, Roots, Fruits, Salt, Coals, Sand, Oyfters, &c.

The ftandard Gallon dry-meafure contains $268\frac{4}{5}$ cubic or folid inches, and the Corn or Winchefter bufhel $2150\frac{2}{5}$ cubic inches ; for the dimenfions of the Winchefter bufhel, by the Statute, are 8 inches deep, and $18\frac{1}{2}$ inches wide or in diameter. But the Coal bufhel muft be $19\frac{1}{2}$ inches in diameter ; and 36 bufhels, heaped up, make a London chaldron of coals, the weight of which is 3156lb Avoirdupois.

## ALE and BEER MEASURE.

| | | | | |
|---|---|---|---|---|
| 2 Pints make | - | 1 Quart | - | Qt |
| 4 Quarts | - - | 1 Gallon | - | Gal |
| 36 Gallons | - | 1 Barrel | - | Bar |
| 1 Barrel and a half | | 1 Hogfhead | | Hhd |
| 2 Barrels | - - | 1 Puncheon | | Pun |
| 2 Hogfheads | - | 1 Butt | - | Butt |
| 2 Butts | - - | 1 Tun | - | Tun |

| Pts | | Qt | | | | |
|---|---|---|---|---|---|---|
| 2 = | | 1 | Gal | | | |
| 8 = | | 4 = | 1 | Bar | | |
| 288 = | | 144 = | 36 = | 1 | Hhd | |
| 432 = | | 216 = | 54 = | $1\frac{1}{2}$ = | 1 | Butt |
| 864 = | | 432 = | 108 = | 3 = | 2 = | 1 |

Note, The Ale Gallon contains 282 cubic or folid Inches.

WINE

## WINE MEASURE.

| | | |
|---|---|---|
| 2 Pints make - - | 1 Quart - - *Qt* |
| 4 Quarts - - | 1 Gallon - - *Gal* |
| 42 Gallons - - | 1 Tierce - - *Tier* |
| 63 Gallons or 1½ Tier | 1 Hogſhead - *Hhd* |
| 2 Tierces - - | 1 Puncheon - *Pun* |
| 2 Hogſheads - - | 1 Pipe or Butt *Pi* |
| 2 Pipes - - | 1 Tun - - *Tun* |

| *Pts* | *Qt* | | | | | | |
|---|---|---|---|---|---|---|---|
| 2 = | 1 | *Gal* | | | | | |
| 8 = | 4 = | 1 | *Tier* | | | | |
| 336 = | 168 = | 42 = | 1 | *Hhd* | | | |
| 504 = | 252 = | 63 = | 1½= | 1 | *Pun* | | |
| 672 = | 336 = | 84 = | 2 = | 1½= | 1 | *Pi* | |
| 1008 = | 504 = | 126 = | 3 = | 2 = | 1½= | 1 | *Tun* |
| 2016 = | 1008 = | 252 = | 6 = | 4 = | 3 = | 2 = | 1 |

*Note,* By this are meaſured all Wines, Spirits, Strong-waters, Cyder, Mead, Perry, Vinegar, Oil, Honey, &c.

The Wine Gallon contains 231 cubic or ſolid inches. And it is remarkable, that the Wine and Ale Gallons have the ſame proportion to each other, as the Troy and Avoirdupois Pounds have ; that is, as one Pound Troy is to one Pound Avoirdupois, ſo is one Wine Gallon to one Ale Gallon.

## *Of* TIME.

| | | |
|---|---|---|
| 60 Seconds or 60″ make | 1 Minute - *M* or ' |
| 60 Minutes - - | 1 Hour - *Hr* |
| 24 Hours - - | 1 Day - *Day* |
| 7 Days - - | 1 Week - *Wk* |
| 4 Weeks - - | 1 Month - *Mo* |
| 13 Months, 1 Day, 6 Hours or 365 Days, 6 Hours } | 1 Julian Year, *Yr* |

| *Sec* | *Min* | | | | | |
|---|---|---|---|---|---|---|
| 60 = | 1 | *Hr* | | | | |
| 3600 = | 60 = | 1 | *Day* | | | |
| 86400 = | 1440 = | 24 = | 1 | *Wk* | | |
| 604800 = | 10080 = | 168 = | 7 = | 1 | *Mo* | |
| 2419200 = | 40320 = | 672 = | 28 = | 4 = | 1 | |
| 31557600 = | 525960 = | 8766 = | 365¼ = | 1 | *Year* | |

*Wk*

> *Wk Da Hr   Mo Da Hr Julian Year*
> Or 52   1   6 = 13   1   6 = 1
> *Da Hr M  Sec*
> But 365   5   48   48 = 1   *Solar Year.*

## RULES for REDUCTION.

I. *When the Numbers are to be reduced from a Higher Denomination to a Lower :*

MULTIPLY the number in the higheſt denomination by as many as of the next lower make an integer, or 1, in that higher; to this product add the number, if any, which was in this lower denomination before, and ſet down the amount.

Reduce this amount in like manner, by multiplying it by as many as of the next lower make an integer of this, taking in the odd parts of this lower, as before. And ſo proceed through all the denominations, to the loweſt; ſo ſhall the number laſt found be the value of all the numbers which were in the higher denominations, taken together*.

### EXAMPLE.

1. In 1234l 15s 7d, how many farthings ?

$$
\begin{array}{ccc}
l & s & d \\
1234 & 15 & 7 \\
20 & & \\
\hline
\end{array}
$$

24695 Shillings
12

296347 Pence
4

Anſwer 1185388 Farthings.

* The reaſon of this rule is very evident; for pounds are brought into ſhillings by multiplying them by 20; ſhillings into pence, by multiplying them by 12; and pence into farthings, by multiplying by 4; and the reverſe of this rule by Diviſion.—And the ſame, it is evident, will be true in the reduction of numbers conſiſting of any denominations whatever.

II. *When*

II. *When the Numbers are to be reduced from a Lower Denomination to a Higher :*

DIVIDE the given number by as many as of that denomination make 1 of the next higher, and set down what remains, as well as the quotient.

Divide the quotient by as many as of this denomination make 1 of the next higher ; setting down the new quotient, and remainder, as before.

Proceed in the same manner through all the denominations, to the highest ; and the quotient last found, together with the several remainders, if any, will be of the same value as the first number proposed.

### EXAMPLES.

2. Reduce 1185388 farthings, into pounds, shillings, and pence.

4 ) 1185388
_____

12 )   296347  d
_____

2,0 )  2469,5  s—7d
_____

Answer ~ 1234 l 15s 7d
_____

3. Reduce 23l to farthings.                 Anf. 22080.
4. Reduce 337587 farthings to pounds, &c.
                                  Anf. 351l 13s 0¾d.
5. How many farthings are in 35 guineas ?   Anf. 35280.
6. In 35280 farthings how many guineas ?       Anf. 35.
7. In 59lb 13 dwts 5 gr how many grains ?  Anf. 340157.
8. In 8012131 grains how many pounds, &c ?
                        Anf. 1390lb. 11oz 18dwt 19gr.
9. In 35 ton 17cwt 1qr 23lb 7oz 13dr how many drams?
                                  Anf. 20571005.
10. How many barley-corns will reach round the earth, supposing it, according to the best calculations, to be 25000 miles ?                      Anf. 4752000000.
11. How many seconds are in a solar year, or 365 days 5hrs 48 min 48 sec?            Anf. 31556928.
12. In a lunar month, or 29ds 12hrs 44min 3sec, how many seconds ?                 Anf 2551443.

COM-

## COMPOUND ADDITION.

COMPOUND ADDITION ſhews how to add or collect ſeveral numbers of different denominations into one ſum.

RULE.—Place the numbers ſo that thoſe of the ſame denomination may ſtand directly under each other, and draw a line below them. Add up the figures in the loweſt denomination, and find, by Reduction, how many units, or ones, of the next higher denomination are contained in their ſum.— Set down the remainder below its proper column, and carry thoſe units or ones to the next denomination, which add up in the ſame manner as before.—Proceed thus through all the denominations, to the higheſt, whoſe ſum, together with the ſeveral remainders, will give the anſwer ſought.

The method of proof is the ſame as in Simple Addition.

### EXAMPLES OF MONEY.

| 1. | | | 2. | | | 3. | | | 4. | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| l | s | d | l | s | d | l | s | d | l | s | d |
| 7 | 13 | 3 | 14 | 7 | 5 | 15 | 17 | 10 | 53 | 14 | 8 |
| 3 | 5 | 10½ | 8 | 19 | 2¼ | 3 | 14 | 6 | 5 | 10 | 2¾ |
| 6 | 18 | 7 | 5 | 3 | 4½ | 23 | 6 | 2¾ | 93 | 11 | 6 |
| 0 | 2 | 5¾ | 21 | 2 | 9 | 8 | 3 | 5 | 7 | 5 | 0 |
| 4 | 0 | 3 | 7 | 16 | 8½ | 15 | 6 | 4 | 2 | 0 | 9 |
| 17 | 15 | 4½ | 0 | 4 | 3 | 6 | 12 | 9¾ | 0 | 18 | 7 |
| 39 | 15 | 9¾ | | | | | | | | | |
| 32 | 2 | 6¾ | | | | | | | | | |
| 39 | 15 | 9¾ | | | | | | | | | |

| 5. | | | 6. | | | 7. | | | 8. | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| l | s | d | l | s | d | l | s | d | l | s | d |
| 14 | 0 | 7¼ | 37 | 15 | 8 | 61 | 3 | 2½ | 472 | 15 | 3 |
| 5 | 13 | 6 | 14 | 12 | 9¾ | 7 | 16 | 8 | 9 | 2 | 2½ |
| 62 | 4 | 7 | 5 | 6 | 11 | 29 | 13 | 10¾ | 27 | 12 | 6¼ |
| 4 | 17 | 8 | 23 | 10 | 9¼ | 8 | 14 | 0 | 370 | 16 | 2½ |
| 23 | 0 | 4¼ | 8 | 6 | 0 | 0 | 7 | 5¼ | 25 | 3 | 8 |
| 6 | 6 | 7 | 14 | 0 | 5½ | 24 | 13 | 0 | 6 | 10 | 5¼ |
| 91 | 0 | 10¼ | 54 | 2 | 7½ | 5 | 0 | 10¾ | 30 | 0 | 11¾ |

EXAM. 9. A nobleman, going out of town, is informed by his fteward, that his butcher's bill comes to 197l 13s 7½d ; his baker's to 59l 5s 2¾d ; his brewer's to 85l; his wine-merchant's to 103l 13s ; to his corn chandler is due 75l 3d ; to his tallow-chandler and cheefe-monger, 27l 15s 11¼d ; and to his tailor, 55l 3s 5¾d ; alfo for rent, fervants wages, and other charges, 127l 3s : Now, fuppofing he would take 100l with him, to defray his charges on the road, for what fum muft he fend to his banker ?        Anf. 830l 14s 6¼d.

10. The ftrength of a regiment of foot, of 10 companies, and the amount of their fubfiftence*, for a month of 30 days, according to the annexed Table, are required ?

| Numb. | Rank. | Subfiftence for a Month. | | |
|---|---|---|---|---|
| | | l | s | d |
| 1 | Colonel | 27 | 0 | 0 |
| 1 | Lieutenant Colonel | 19 | 10 | 0 |
| 1 | Major | 17 | 5 | 0 |
| 7 | Captains | 78 | 15 | 0 |
| 11 | Lieutenants | 57 | 15 | 0 |
| 9 | Enfigns | 40 | 10 | 0 |
| 1 | Chaplain | 7 | 10 | 0 |
| 1 | Adjutant | 4 | 10 | 0 |
| 1 | Quarter-mafter | 5 | 5 | 0 |
| 1 | Surgeon | 4 | 10 | 0 |
| 1 | Surgeon's Mate | 4 | 10 | 0 |
| 30 | Serjeants | 45 | 0 | 0 |
| 30 | Corporals | 30 | 0 | 0 |
| 20 | Drummers | 20 | 0 | 0 |
| 2 | Fifes | 2 | 0 | 0 |
| 390 | Private men | 292 | 10 | 0 |
| 50 7 | Total. | 656 | 10 | 0 |

* Subfiftence Money, is the money paid to the foldiers weekly; which is fhort of their full pay, becaufe their clothes, accoutrements, &c. are to be accounted for. It is likewife the money advanced to officers till their accounts are made up, which is commonly once a year, when they are paid their arrears. The following Table fhews the full pay and fubfiftence of each rank on the Englifh eftablifhment:

DAILY PAY OF COMMISSIONED OFFICERS, ACCORDING TO THE LATEST REGULATIONS.

| RANK | Horse Artillery & corps of cap. com. | D. G. D. & F. Cav. | Foot Ar. tillery. | Regular and Fenc. Inf. and Militia. | Life Guards Subsist. | Life Guards Full Pay. | Horse Guards Susbist. | Horse Guards Full Pay. | Foot Guards Subsist. | Foot Guards Full Pay. |
|---|---|---|---|---|---|---|---|---|---|---|
| Colonel (Comm.) | — | 1 12 10 | 2 4 0 | 1 2 6 | 1 7 0 | 1 16 0 | 1 11 0 | — | 1 10 0 | 1 19 0 |
| Colonel (en Second) | 1 10 0 | — | 1 4 0 | — | — | — | — | — | — | — |
| 1st Lieut. Col. | 1 6 0 | 1 3 0 | 1 0 0 | 0 15 11 | 1 3 0 | 1 11 0 | 1 2 6 | — | 1 1 6 | 1 8 6 |
| 2d Ditto | — | — | 0 17 0 | — | — | — | — | — | — | — |
| 1st Major | 1 0 0 | 0 19 0 | 0 15 0 | 0 14 1 | 0 19 6 | 1 6 0 | 1 6 | — | 0 18 6 | 1 4 6 |
| 2d Ditto | — | — | — | — | 0 18 0 | 1 4 0 | — | — | — | — |
| Captain | 0 15 0 | 0 14 7 | 0 10 0 | 0 9 5 | 0 12 10 | 0 16 0 | 0 16 6 | 0 15 0 | 0 12 6 | 0 16 6 |
| Capt. Lieut. | 0 10 0 | 0 9 0 | 0 7 0 | 0 5 8 | 0 8 0 | 0 11 0 | 0 11 6 | — | 0 6 0 | 0 7 10 |
| 1st Lieut. | 0 9 0 | 0 9 0 | 0 6 0 | 0 5 8 | — | — | — | — | — | — |
| 2d Ditto | 0 8 0 | — | 0 5 0 | — | — | — | — | — | — | — |
| Cornet | — | 0 8 0 | — | — | — | — | 0 11 0 | 0 14 0 | 0 4 6 | 0 5 10 |
| Ensign | — | — | — | 0 4 8 | — | — | — | — | 0 3 0 | 0 4 0 |
| Adjutant | — | 0 5 0 | — | 0 5 0 | 0 8 6 | 0 11 0 | 0 4 6 | 0 5 0 | — | — |
| Pay-master | — | — | — | — | — | — | — | — | — | — |
| Quarter-master | — | 0 5 0 | — | 0 5 8 | — | — | — | — | — | 0 5 8 |
| Surgeon-major | 0 12 0 | 0 11 4 | 0 10 0 | 0 9 5 | 0 6 0 | 8 6 | 0 6 6 | 8 6 | 0 12 6 | 0 15 0 |
| Bat. Surg. or Surg. | — | — | 0 5 0 | — | 0 6 0 | 0 8 0 | — | — | 0 7 0 | 0 10 0 |
| Assist. Surg. | 0 6 0 | 0 5 0 | 0 5 0 | 0 5 0 | 0 9 0 | 0 12 0 | 0 9 0 | 0 12 0 | 0 5 0 | 0 5 0 |
| Veter. Surg. | 0 8 0 | 0 8 0 | — | — | — | — | 0 5 0 | 0 5 0 | — | — |
| Solicitor | — | — | — | — | — | — | — | — | — | — |

N. B. When a *Lieutenant*, *Ensign*, *Adjutant*, or *Quarter-master* of Foot, Militia, Fencible Infantry, or Invalids, holds two commissions, one shilling per day is to be deducted from the above rates for each commission.

## EXAMPLES of WEIGHTS, MEASURES, &c.

### TROY WEIGHT.   APOTHECARIES WEIGHT.

| 1. | | | 2. | | | 3. | | | | 4. | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| lb | oz | dwt | oz | dwt | gr | lb | oz | dr | ſc | oz | dr | ſc | gr |
| 17 | 3 | 15 | 37 | 9 | 3 | 3 | 5 | 7 | 2 | 3 | 5 | 1 | 17 |
| 4 | 6 | 3 | 9 | 5 | 3 | 13 | 7 | 3 | 0 | 7 | 3 | 2 | 5 |
| 0 | 10 | 7 | 3 | 16 | 21 | 9 | 11 | 0 | 1 | 16 | 7 | 0 | 12 |
| 9 | 5 | 0 | 17 | 7 | 8 | 0 | 9 | 1 | 2 | 9 | 5 | 1 | 5 |
| 176 | 2 | 17 | 5 | 9 | 0 | 36 | 3 | 5 | 0 | 4 | 1 | 2 | 18 |
| 23 | 11 | 12 | 3 | 0 | 19 | 5 | 8 | 6 | 1 | 36 | 4 | 1 | 14 |

### AVOIRDUPOIS WEIGHT.   LONG MEASURE.

| 5. | | | 6. | | | 7. | | | 8. | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| lb | oz | dr | cwt | qr | lb | mls | fur | pls | yds | feet | inc |
| 17 | 10 | 13 | 15 | 2 | 15 | 29 | 3 | 14 | 127 | 1 | 5 |
| 5 | 14 | 8 | 6 | 3 | 24 | 19 | 6 | 29 | 12 | 2 | 9 |
| 8 | 6 | 15 | 7 | 0 | 10 | 5 | 4 | 20 | 0 | 2 | 6 |
| 27 | 1 | 6 | 9 | 1 | 17 | 9 | 1 | 37 | 54 | 1 | 11 |
| 0 | 4 | 0 | 10 | 2 | 6 | 7 | 0 | 3 | 5 | 2 | 7 |
| 6 | 14 | 10 | 3 | 0 | 3 | 4 | 5 | 9 | 23 | 0 | 5 |

### CLOTH MEASURE.   LAND MEASURE.

| 9. | | | 10. | | | | 11. | | | 12. | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| yds | qr | nls | el | en | qrs | nls | ac | ro | p | ac | ro | p |
| 26 | 3 | 1 | 270 | 1 | 0 | | 225 | 3 | 37 | 19 | 0 | 16 |
| 13 | 1 | 2 | 57 | 4 | 3 | | 16 | 1 | 25 | 270 | 3 | 29 |
| 6 | 2 | 0 | 8 | 2 | 1 | | 9 | 0 | 13 | 9 | 1 | 3 |
| 217 | 0 | 3 | 0 | 3 | 2 | | 4 | 2 | 9 | 23 | 0 | 34 |
| 9 | 1 | 0 | 10 | 1 | 0 | | 42 | 1 | 19 | 7 | 2 | 16 |
| 55 | 3 | 1 | 4 | 4 | 1 | | 7 | 0 | 6 | 75 | 0 | 23 |

### WINE MEASURE.   ALE and BEER MEASURE.

| 13. | | | 14. | | | 15. | | | 16. | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| t | hds | gal | hds | gal | pts | hds | gal | pts | hds | gal | pts |
| 13 | 3 | 15 | 15 | 61 | 5 | 17 | 37 | 3 | 29 | 43 | 5 |
| 8 | 1 | 37 | 7 | 16 | 3 | 4 | 13 | 5 | 7 | 9 | 2 |
| 4 | 2 | 26 | 29 | 23 | 7 | 3 | 6 | 2 | 14 | 16 | 6 |
| 25 | 0 | 12 | 3 | 15 | 1 | 5 | 14 | 0 | 6 | 8 | 1 |
| 3 | 1 | 9 | 16 | 8 | 0 | 12 | 9 | 6 | 57 | 13 | 4 |
| 72 | 3 | 21 | 4 | 36 | 6 | 8 | 42 | 4 | 5 | 6 | 0 |

## COMPOUND SUBTRACTION.

COMPOUND SUBTRACTION fhews how to find the dif-
ference between any two numbers of different denomina-
tions.—To perform which, obferve the following Rule :

### RULE. *

PLACE the lefs number below the greater, fo that thofe
parts which are of the fame denomination may ftand directly
under each other ; and draw a line below them.—Begin at
the right-hand, and fubtract each number or part in the lower
line, from the one juft above it, and fet the remainder ftraight
below it.—But if any number in the lower line be greater
than that above it, add fo many to the upper number as
make 1 of the next higher denomination ; then take the
lower number from the upper one thus increafed, and fet
down the remainder.  Carry the unit borrowed to the next
number in the lower line ; after which fubtract this number
from the one above it, as before ; and fo proceed till the whole
is finifhed.  Then the feveral remainders, taken together,
will be the whole difference fought.

The method of Proof is the fame as in Simple Subtraction.

### EXAMPLES OF MONEY.

|         | 1. | | | 2. | | | 3. | | | 4. | | |
|---------|----|---|---|----|---|---|----|---|---|-----|----|----|
|         | l  | s | d | l  | s | d | l  | s | d | l   | s  | d  |
| From    | 79 | 17 | $8\frac{3}{4}$ | 103 | 3 | $2\frac{1}{2}$ | 57 | 0 | 10 | 251 | 13 | 0 |
| Take    | 35 | 12 | $4\frac{1}{4}$ | 71 | 12 | $5\frac{3}{4}$ | 29 | 13 | $3\frac{1}{4}$ | 35 | 4 | $7\frac{3}{4}$ |
| Rem.    | 44 | 5 | $4\frac{1}{2}$ | 31 | 10 | $8\frac{3}{4}$ | | | | | | |
| Proof   | 79 | 17 | $8\frac{3}{4}$ | 103 | 3 | $2\frac{1}{2}$ | | | | | | |

5. What is the difference between 73l 5¼d and 19l 13s 10d ?
Anf. 53l 6s 7¼d..

---

* The reafon of this Rule will eafily appear from what has been
faid in Simple Subtraction; for the borrowing depends on the
fame principle, and is only different as the numbers to be fubtracted
are of different denominations.

Ex. 6.

Ex. 6. A lends to B 100l, how much is B in debt after A has taken goods of him to the amount of 73l 12s 4¾d?

Anf. 26l 7s 7¼d.

7. Suppofe that my rent for half a year is 20l 12s, and that I have laid out for the land-tax 14s 6d, and for feveral repairs 1l 3s 3¼d, what have I to pay of my half year's rent?

Anf. 18l 14s 2¾d.

8. A trader, failing, owes to A 35l 7s 6d, to B 91l 13s ½d, to C 53l 7¼d, to D 87l 5s, and to E 111l 3s 5¾d. When this happened, he had by him in cafh, 23l 7s 5d, in wares 53l 11s 10¼d, in houfehold furniture 63l 17s 7¾d, and in recoverable book-debts 25l 7s 5d. What will his creditors lofe by him, fuppofe thefe things delivered to them?

Anf. 212l 5s 6½d.

### EXAMPLES of WEIGHTS, MEASURES, &c.

| | TROY WEIGHT. 1. | | | | APOTHECAR. WEIGHT. 3. | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | lb | oz | dwt | gr | lb | oz | dr | fcr | gr |
| From | 7 | 3 | 14 | 11 | 73 | 4 | 7 | 0 | 14 |
| Take | 3 | 7 | 5 | 19 | 26 | 7 | 2 | 1 | 16 |
| Rem. | | | | | | | | | |
| Proof | | | | | | | | | |

| | 2. | | | |
|---|---|---|---|---|
| | lb | oz | dwt | gr |
| From | 4 | 9 | 1 | 13 |
| Take | 3 | 7 | 16 | 12 |

| | AVOIRDUPOIS WEIGHT. 4. | | | 5. | | | LONG MEASURE. 6. | | | 7. | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | c | qrs | lb | lb | oz | dr | m | fu | pl | yd | ft | in |
| From | 5 | 0 | 17 | 71 | 5 | 9 | 14 | 3 | 17 | 96 | 1 | 4 |
| Take | 3 | 2 | 11 | 14 | 6 | 14 | 3 | 7 | 9 | 41 | 2 | 7 |
| Rem. | | | | | | | | | | | | |
| Proof | | | | | | | | | | | | |

| | CLOTH MEASURE. 8. | | | 9. | | | LAND MEASURE. 10. | | | 11. | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | yd | qr | nl | yd | qr | nl | ac | ro | p | ac | ro | p |
| From | 17 | 2 | 1 | 9 | 0 | 2 | 17 | 1 | 14 | 57 | 1 | 16 |
| Take | 5 | 2 | 1 | 6 | 1 | 2 | 6 | 3 | 6 | 24 | 2 | 25 |
| Rem. | | | | | | | | | | | | |
| Proof | | | | | | | | | | | | |

|  | WINE MEASURE. |  |  |  |  | ALE & BEER MEASURE. |  |  |  |  |
|  | 12. |  |  | 13. |  |  | 14. |  |  | 15. |  |
|  | t | hd | gal | hd | gal | pt | hd | gal | pt | hd | gal | pt |
| From | 17 | 2 | 23 | 5 | 0 | 4 | 14 | 29 | 3 | 71 | 16 | 5 |
| Take | 4 | 3 | 39 | 3 | 2 | 7 | 7 | 34 | 5 | 17 | 3 | 2 |
| Rem. | | | | | | | | | | | | |
| Proof | | | | | | | | | | | | |

|  | DRY MEASURE. |  |  |  |  | TIME |  |  |  |  |
|  | 16. |  |  | 17. |  | 18. |  |  | 19. |  |  |
|  | la | qr | bu | bu | gal | pt | mo | we | da | ds | hrs | min |
| From | 9 | 4 | 7 | 13 | 7 | 1 | 71 | 2 | 5 | 114 | 17 | 26 |
| Take | 3 | 7 | 2 | 7 | 3 | 4 | 14 | 3 | 0 | 75 | 12 | 35 |
| Rem. | | | | | | | | | | | | |
| Proof | | | | | | | | | | | | |

20. The line of defence in a certain polygon being 236 yards, and that part of it which is terminated by the curtain and fhoulder being 146 yards 1 foot 4 inches; what then was the length of the face of the baftion ?  Anf. 89 yds 1 f 8 in.

## COMPOUND MULTIPLICATION.

COMPOUND MULTIPLICATION fhews how to find the amount of any given number of different denominations repeated a certain propofed number of times.

### RULE.

SET the multiplier under the loweft denomination of the multiplicand, and draw a line below it.—Multiply the number in the loweft denomination by the multiplier, and find how many units of the next higher denomination are contained in the product, fetting down what remains.—In like manner multiply the number in the next denomination, and to the product carry or add the units, before found, and find how many units of the next higher denomination are in this amount,

amount, which carry in like manner to the next product, fetting down the overplus.—Proceed thus to the higheft denomination propofed ; fo fhall the laft product, with the feveral remainders, taken as one compound number, be the whole amount required.

The method of Proof, and the reafon of the Rule, are the fame as in Simple Multiplication.

### EXAMPLES OF MONEY.

1. To find the amount of 8lb of Tea, at 5s 8d½ per lb.

```
        s    d
        5    8½
             8
    _____
    £2   5    8  Anfwer.
```

|    |                                          |      | l  | s  | d    |
|----|------------------------------------------|------|----|----|------|
| 2. | 4lb of Tea, at 7s 8d per lb.             | Anf. | 1  | 10 | 8    |
| 3. | 6lb of Butter, at 9½d per lb.            | Anf. | 0  | 4  | 9    |
| 4. | 7lb of Tobacco, at 1s 8½d per lb.        | Anf. | 0  | 11 | 11½  |
| 5. | 9 cwt of Cheefe, at 1l 11s 5d per cwt.   | Anf. | 14 | 2  | 9    |
| 6. | 10 cwt of Cheefe, at 2l 17s 10d per cwt. | Anf. | 28 | 18 | 4    |
| 7. | 12 cwt of Sugar, at 3l 7s 4d per cwt.    | Anf. | 40 | 8  | 0    |

### CONTRACTIONS.

I. If the multiplier exceed 12, multiply fucceffively by its component parts, inftead of the whole number at once.

### EXAMPLES.

1. 15 cwt of Cheefe, at 17s 6d per cwt.

```
     l    s    d
     0    17   6
               3
    _____
     2    12   6
               5
    _____
    13    2    6  Anfwer.
```

|    |                                       |      | l  | s | d |
|----|---------------------------------------|------|----|---|---|
| 2. | 20 cwt of Hops, at 4l 7s 2d per cwt.  | Anf. | 87 | 3 | 4 |
| 3. | 24 tons of Hay, at 3l 7s 6d per ton.  | Anf. | 81 | 0 | 0 |
| 4. | 45 ells of Cloth, at 1s 6d per ell.   | Anf. | 3  | 7 | 6 |

Ex. 5.

                                l   s   d

Ex. 5. 63 gallons of Oil, at 2s 3d per gall. Anf.   7   1   9

    6. 70 barrels of Ale, at 1l 4s per barrel. Anf.  84   0   0

    7. 84 quarters of Oats, at 1l 12s 8d per qr. Anf. 137   4   0

    8. 96 quarters of Barley, at 1l 3s 4d per qr. Anf. 112   0   0

    9. 120 days Wages, at 5s 9d per day.  Anf. 34  10   0

   10. 144 reams of Paper, at 13s 4d per ream. Anf. 96   0   0

II. If the multiplier cannot be exactly produced by the multiplication of fimple numbers, take the neareft number to it, either greater or lefs, which can be fo produced, and multiply by its parts, as before.—Then multiply the given multiplicand by the difference between this affumed number and the multiplier, and add the product to that before found when the affumed number is lefs than the multiplier, but fubtract the fame when it is greater.

EXAMPLES.

1. 26 yards of Cloth, at 3s 0¾d per yard.

$$
\begin{array}{ccc}
\text{l} & \text{s} & \text{d} \\
0 & 3 & 0\frac{3}{4} \\
 & & 5 \\
\hline
0 & 15 & 3\frac{3}{4} \\
 & & 5 \\
\hline
3 & 16 & 6\frac{3}{4} \\
 & 3 & 0\frac{3}{4} \\
\hline
£3 & 19 & 7\frac{1}{2}
\end{array}
$$ Anfwer.

                                l   s   d

2. 29 quarters of Corn, at 2l 5s 3¼d per qr. Anf. 65  12  10¼

3. 53 loads of Hay, at 3l 15s 2d per load.  Anf. 199   3  10

4. 79 bufhels of Wheat, at 11s 5¾d per bufh. Anf.  45   6  10¼

5. 94 cafks of Beer, at 12s 2d per cafk.  Anf.  57   3   8

6. 114 ftone of Meat, at 15s 3¾d per ftone. Anf. 87   5   7½

EXAMPLES OF WEIGHTS AND MEASURES.

| 1. | | | | 2. | | | | | 3. | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| lb | oz | dwt | gr | lb | oz | dr | fc | gr | cwt | qr | lb | oz |
| 21 | 1 | 7 | 13 | 2 | 4 | 2 | 1 | 0 | 27 | 1 | 13 | 12 |
| | | | 4 | | | | | 7 | | | | 12 |

| 4. | | | | 5. | | | 6. | | |
|---|---|---|---|---|---|---|---|---|---|
| mls | fu | pls | yds | yds | qrs | na | ac | ro | po |
| 24 | 3 | 20 | 2 | 127 | 2 | 2 | 27 | 2 | 1 |
| | | | 6 | | | 8 | | | 9 |

| 7. | | | | 8. | | | | 9. | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| tuns | hhd | gal | pts | we | qr | bu | pe | mo | we | da | ho | min |
| 29 | 1 | 20 | 3 | 27 | 1 | 7 | 2 | 175 | 3 | 6 | 20 | 59 |
| | | | 5 | | | | 7 | | | | | 11 |

# COMPOUND DIVISION.

COMPOUND DIVISION teaches how to divide a number of feveral denominations by any given number, or into any number of equal parts.

## R U L E.

PLACE the divifor on the left of the dividend, as in Simple Divifion.—Begin at the left-hand, and divide the number of the higheft denomination by the divifor, fetting down the quotient in its proper place.—If there be any remainder after this divifion, reduce it to the next lower denomination, which add to the number, if any, belonging to that denomination, and divide the fum by the divifor.—Set down again this quotient, reduce its remainder to the next lower denomination again, and fo on through all the denominations to the laft.

### EXAMPLES OF MONEY.

1. Divide 225l 2s 4d by 2.

```
         l    s   d
   2 )  225   2   4
       ───────────────
     £112   11   2  the Quotient.
       ───────────────
```

2. Di-

|  | l | s | d |  |  | l | s | d |
|--|---|---|---|--|--|---|---|---|
| 2. Divide | 751 | 14 | 7¾ | by 3. | Anſ. | 250 | 11 | 6½ |
| 3. Divide | 821 | 17 | 9¾ | by 4. | Anſ. | 205 | 9 | 5¼ |
| 4. Divide | 2382 | 13 | 5½ | by 5. | Anſ. | 476 | 10 | 8¼ |
| 5. Divide | 28 | 2 | 1½ | by 6. | Anſ. | 4 | 13 | 8¼ |
| 6. Divide | 55 | 14 | 0¾ | by 7. | Anſ. | 7 | 19 | 1¾ |
| 7. Divide | 6 | 5 | 4 | by 8. | Anſ. | 0 | 15 | 8 |
| 8. Divide | 135 | 10 | 7 | by 9. | Anſ. | 15 | 1 | 2 |
| 9. Divide | 21 | 18 | 4 | by 10. | Anſ. | 2 | 3 | 10 |
| 10. Divide | 227 | 10 | 5 | by 11. | Anſ. | 20 | 13 | 8 |
| 11. Divide | 1332 | 11 | 8½ | by 12. | Anſ. | 111 | 0 | 11½ |

## CONTRACTIONS.

I. If the diviſor exceed 12, find what ſimple numbers, multiplied together, will produce it, and divide by them ſeparately, as in Simple Diviſion.

### EXAMPLES.

1. What is Cheeſe per cwt if 16 cwt coſt 30l 18s 8d ?

$$
\begin{array}{r}
\phantom{4)}\ \ l\ \ \ s\ \ \ d \\
4)\ 30\ \ 18\ \ 8 \\
\hline
4)\ 7\ \ 14\ \ 8 \\
\hline
£\ \ 1\ \ 18\ \ 8 \ \text{the Anſwer.} \\
\hline
\end{array}
$$

|  |  | l | s | d |
|--|--|---|---|---|
| 2. If 20 cwt of Tobacco come to 120l 10s, what is that per cwt? | Anſ. | 6 | 0 | 6 |
| 3. Divide 57l 3s 7d by 35. | Anſ. | 1 | 12 | 8 |
| 4. Divide 85l 6s by 72. | Anſ. | 1 | 3 | 8¼ |
| 5. Divide 31l 2s 10½d by 99. | Anſ. | 0 | 6 | 3½ |
| 6. At 18l 18s per cwt, how much per lb ? | Anſ. | 0 | 3 | 4½ |

II. If the diviſor cannot be produced by the multiplication of ſmall numbers, divide by the whole diviſor at once, after the manner of Long Diviſion.

EXAMPLES.

1. Divide 74l 13s 6d by 17.

```
         l    s   d        l  s   d
17 ) 74    13   6    ( 4  7  10  Anfwer.
     68
     ──
      6
     20
     ──
    133
    119
    ───
     14
     12
    ───
    174
    170
    ───
      4
```

|   | l | s | d |   |   |   | l | s | d |
|---|---|---|---|---|---|---|---|---|---|
| 2. Divide | 23 | 15 | 7¼ by | 37. | Anf. | 0 | 12 | 10¼ |
| 3. Divide | 199 | 3 | 10 by | 53. | Anf. | 3 | 15 | 2 |
| 4. Divide | 675 | 12 | 6 by | 138. | Anf. | 4 | 17 | 11 |
| 5. Divide | 315 | 3 | 10¼ by | 365. | Anf. | 0 | 17 | 3¼ |

EXAMPLES OF WEIGHTS AND MEASURES.

1. Divide 23lb 7oz 6dwts 12gr by 7.

Anf. 3lb 4oz 9dwts 12gr.

2. Divide 13lb 1oz 2dr ofcr 10gr by 12.

Anf. 1lb 1oz 0dr 2fcr 10gr.

3. Divide 1061cwt 2qrs by 28. Anf. 37cwt 3 qrs 18lb.

4. Divide 375mi 2fur 7po 2yds 1ft 2in by 39.

Anf. 9mi 4fur 39po 0yds 2ft 8in.

5. Divide 571yds 2qrs 1na by 47. Anf. 12yd 0qr 2na.

6. Divide 51ac 2ro 3po by 51. Anf. 1ac 0ro 1po.

7. Divide 10tu 2hhds 17gall 2pi by 67. Anf. 39gal 3p.

8. Divide 120la 1qr 1bu 2pe by 74. Anf. 1la 6qrs 1bu 3pe.

9. Divide 120mo 2we 3da 5ho 20min by 111.

Anf. 1mo 0we 2da 10ho 12mi.

GOLDEN

## GOLDEN RULE, or RULE OF THREE.

THE RULE OF THREE teaches how to find a fourth proportional to three numbers given: for which reason it is sometimes called the Rule of Proportion. It is called the Rule of Three, becaufe three terms or numbers are given, to find a fourth. And becaufe of its great and extenfive ufefulnefs, it is often called the Golden Rule.

This Rule is ufually confidered as of two kinds, namely, Direct, and Inverfe.

The Rule of Three Direct, is that in which more requires more, or lefs requires lefs. As in this, if 3 men dig 21 yards of trench in a certain time, how much will 6 men dig in the fame time? Here more requires more, that is, 6 men, which are more than 3 men, will alfo perform more work, in the fame time. Or when it is thus: if 6 men dig 42 yards, how much will 3 men dig in the fame time? Here, then, lefs requires lefs, or 3 men will perform proportionally lefs work than 6 men, in the fame time. In both thefe cafes, then, the Rule, or the Proportion, is Direct: and the ftating muft be

thus, As 3 : 21 :: 6 : 42,
or thus, As 6 : 42 :: 3 : 21.

But the Rule of Three Inverfe, is when more requires lefs, or lefs requires more. As in this; if 3 men dig a certain quantity of trench in 14 hours, in how many hours will 6 men dig the like quantity? Here it is evident that 6 men, being more than 3, will perform the equal quantity of work in lefs time, or fewer hours. Or thus; if 6 men perform a certain quantity of work in 7 hours, in how many hours will 3 men perform the fame? Here lefs requires more, for 3 men will take more hours than 6 to perform the fame work. In both thefe cafes, then, the Rule, or the Proportion, is Inverfe; and the ftating muft be

thus, As 6 : 14 :: 3 : 7,
or thus, As 3 : 7 :: 6 : 14.

And in all thefe ftatings, the fourth term is found, by multiplying the 2d and 3d terms together, and dividing the product by the 1ft term.

Of the three given numbers; two of them contain the fuppofition, and the third a demand. And for ftating and working queftions of thefe kinds, obferve the following general Rule:

RULE.

## RULE.

State the queſtion, by ſetting down in a ſtraight line the three given numbers, in the following manner, viz. ſo that the 2d term be that number of ſuppoſition which is of the ſame kind that the anſwer or 4th term is to be ; making the other number of ſuppoſition the 1ſt term, and the demanding number the 3d term, when the queſtion is in direct proportion ; but contrariwiſe, the other number of ſuppoſition the 3d term, and the demanding number the 1ſt term, when the queſtion has inverſe proportion.

Then, in both caſes, multiply the 2d and 3d terms together, and divide the product by the 1ſt, which will give the anſwer, or 4th term ſought, of the ſame denomination as the ſecond term.

*Note,* If the firſt and third terms conſiſt of different denominations, reduce them both to the ſame : and if the ſecond term be a compound number, it is moſtly convenient to reduce it to the loweſt denomination mentioned.—If, after diviſion, there be any remainder, reduce it to the next lower denomination, and divide by the ſame diviſor as before, and the quotient will be of this laſt denomination. Proceed in the ſame manner with all the remainders, till they be reduced to the loweſt denomination which the ſecond term admits of, and the ſeveral quotients taken together will be the anſwer required.

*Note* alſo, The reaſon for the foregoing Rules will appear when we come to treat of the nature of Proportions.—Sometimes alſo two or more ſtatings are neceſſary, which may always be known from the nature of the queſtion.

### EXAMPLES.

1. If 8 yards of Cloth coſt 1l 4s, what will 96 yards coſt ?

$$\begin{array}{cccccc} \text{yds} & l & s & \text{yds} & l & s \\ \text{As } 8 & : & 4 & :: 96 & : & 14 \ 8 \ \text{the Anſwer.} \end{array}$$

$$\begin{array}{r} 20 \\ \hline 24 \\ 96 \\ \hline 144 \\ 216 \\ \hline 8\ )\ 2304 \\ 2,0\ )\ 28,8s \\ \hline £14\ 8\ \text{Anſwer.} \end{array}$$

Ex. 2. An engineer having raifed 100 yards of a certain work in 24 days with 5 men ; how many men muft he employ to finifh a like quantity of work in 15 days ?

$$\text{As} \quad \underset{15}{\overset{ds}{\phantom{1}}} : \underset{5}{\overset{men}{\phantom{1}}} :: \underset{24}{\overset{ds}{\phantom{1}}} : \underset{8}{\overset{men}{\phantom{1}}} \text{ Anf.}$$

$$\begin{array}{r} 5 \\ \hline 15 ) 120 ( 8 \text{ Anfwer.} \\ 120 \\ \hline \end{array}$$

3. What will 72 yards of Cloth coft, at the rate of 9 yards for 5l 12s ? Anf. 44l 16s.

4. A perfon's annual income being 146l ; how much is that per day ? Anf. 8s.

5. If 3 paces or common fteps of a certain perfon be equal to 2 yards ; how many yards will 160 of his paces make ? Anf. 106yds 2ft.

6. What length muft be cut off a board, that is 9 inches broad, to make a fquare foot, or as much as 12 inches in length and 12 in breadth contains ? Anf. 16 inches.

7. If 750 men require 22500 rations of bread for a month ; how many rations will a garrifon of 1200 men require ? Anf. 36000.

8. If 7cwt 1qr of fugar coft 26l 10s 4d, what will be the price of 43cwt 2qrs ? Anf. 159l 2s.

9. The clothing of a regiment of foot of 750 men amounting to 2831l 5s ; what will the clothing of a body of 3500 men amount to ? Anf. 13212l 10s.

10. How many yards of matting, that is 2ft 6in broad, will cover a floor that is 27 feet long and 20 feet broad ? Anf. 72yds.

11. What is the value of 6 bufhels of coals, at the rate of 1l 14s 6d the chaldron ? Anf. 5s 9d.

12. If 6352 ftones of 3 feet long complete a certain quantity of walling ; how many ftones of 2 feet long will raife a like quantity ? Anf. 9528.

13. What muft be given for a piece of filver, weighing 73lb 5oz 15dwts, at the rate of 5s 9d per ounce ? Anf. 253l 10s 0$\frac{3}{4}$d.

14. A garrifon of 536 men having provifion for 12 months ; how long will thofe provifions laft, if the garrifon be increafed to 1124 men ? Anf. 174 days and $\frac{64}{1124}$.

15. What will the tax upon 763l 15s be, at the rate of 3s 6d per pound fterling ? Anf. 133l 13s 1$\frac{1}{2}$d.

16. A

16. A certain work being raifed in 12 days, by working 4 hours each day ; how long would it have been in raifing by working 6 hours per day ? Anf. 8 days.

17. What quantity of corn can I buy for 90 guineas, at the rate of 6s the bufhel ? Anf. 39qrs 3bu.

18. A perfon, failing in trade, owes in all 977l ; at which time he has in money, goods, and recoverable debts, 420l 6s 3¼d ; now fuppofing thefe things delivered to his creditors, how much will they get per pound ? Anf. 8s 7¼d.

19. A plain of a certain extent having fupplied a body of 3000 horfe with forage for 18 days ; then how many days would the fame plain have fupplied a body of 2000 horfe ? Anf. 27 days.

20. Suppofe a gentleman's income is 500 guineas a year, and that he fpends 19s 7d per day, one day with another ; how much will he have faved at the year's end ? Anf. 167l 12s 1d.

21. What coft 30 pieces of lead, each weighing 1cwt 12lb, at the rate of 16s 4d the cwt ? Anf. 27l 2s 6d

22. The governor of a befieged place having provifion for 54 days, at the rate of 1½lb of bread ; but being defirous to prolong the fiege to 80 days, in expectation of fuccour, in that cafe what muft the ration of bread be ? Anf. 1$\frac{1}{80}$lb.

23. At half a guinea per week, how long can I be boarded for 20 pounds ? Anf. 38$\frac{12}{126}$wks.

24. How much will 75 chaldrons 7 bufhels of coals come to, at the rate of 1l 13s 6d per chaldron ? Anf. 125l 19s 1⅙d.

25. If the penny loaf weigh 9 ounces when the bufhel of wheat coft 6s 3d; what ought the penny loaf to weigh when the wheat is at 8s 2½d ? Anf. 6oz 13$\frac{122}{197}$dr.

26. How much a year will 173 acres 2 roods 14 poles of land give, at the rate of 1l 7s 8d per acre ? Anf. 240l 2s 7$\frac{1}{20}$d.

27. To how much amounts 172 pieces of lead, each weighing 3cwt 2qrs 17½lb, at 8l 17s 6d per fother of 19½cwt ? Anf. 286l 4s 4½d.

28. How many yards of ftuff, of 3qrs wide, will line a cloak that is 5½ yards in length and 1¼ yard wide ? Anf. 9yds 2qrs 2⅔nl.

29. If 5 yards of cloth coft 14s 2d, what muft be given for 9 pieces, containing each 21 yards 1 quarter ? Anf. 27l 1s 10½d.

30. If a gentleman's eftate be worth 2107l 12s a year ; what may he fpend per day, to fave 500l in the year ? Anf. 4l 8s 1$\frac{19}{365}$d.

31. Want-

31. Wanting juft an acre of land cut off from a piece which is 13½ poles in breadth, what length muft the piece be ?

Anf. 11po 4yds 2ft 0½⁸⁄₇in.

32. At 13s 2½d per yard, what is the value of a piece of cloth containing 52¾ ells Englifh ?        Anf. 43l 10s 11⁶⁄₁₆d.

33. If the carriage of 5cwt 14lb for 96 miles be 1l 12s 6d ; how far may I have 3cwt 1qr carried for the fame money ?

Anf. 151m 3fur 3¹⁄₁₃pol.

34. Bought a filver tankard, weighing 1lb 7oz 14dwts ; what did it coft me at 6s 4d the ounce ?        Anf. 6l 4s 9⅕d.

35. What is the half year's rent of 547 acres of land, at 15s 6d the acre ?                Anf. 21 1l 19s 3d.

36. A wall that is to be built to the height of 27 feet, was raifed 9 feet high by 12 men in 6 days ; then how many men muft be employed to finifh the wall in 4 days, at the fame rate of working ?                Anf. 36 men.

37. What will be the charge of keeping 11 horfes for a year, at the rate of 11½d per day for each horfe ?

Anf. 192l 7s 8½d.

38. If 15 ells of ftuff that is ¾ yard wide coft 37s 6d ; what will 40 ells, of the fame goodnefs, coft, being yard wide ?

Anf. 6l 13s 4d.

39. How many yards of paper that is 30 inches wide, will hang a room that is 20 yards in circuit, and 9 feet high ?

Anf. 72 yds.

40. If a gentleman's eftate be worth 384l 16s a year, and the land-tax be affeffed at 2s 9½d per pound, what is his net annual income ?                Anf. 331l 1 s 9¾d.

41. The circumference of the earth is about 25000 miles ; at what rate per hour is a perfon at the middle of its furface carried round, one whole rotation being made in 23 hours 56 minutes ?                Anf. 1044⁸¹⁶⁄₁₄₃₆ miles.

42. If a perfon drink 20 bottles of wine per month, when it cofts 8s a gallon ; how many bottles per month may he drink, without increafing the expence, when wine cofts 10s the gallon ?                Anf. 16 bottles.

43. What coft 43qrs 5 bufhels of corn, at 1l 8s 6d the quarter ?                Anf. 62l 3s 3¾d.

44. How many yards of canvas that is ell wide, will line 20 yards of fay that is 3 quarters wide ?        Anf. 12yds.

45. If an ounce of gold coft 4 guineas, what is the value of a grain ?                Anf. 2¹⁄₁₀d.

46. If 3cwt of tea coft 40l 12s ; at how much a pound muft it be retailed, to gain 10l by the whole ?    Anf. 3³⁴⁄₃₆s.

COMPOUND

# COMPOUND PROPORTION.

Compound Proportion teaches how to refolve fuch queftions as require two or more ftatings by Simple Proportion : and that, whether they be Direct or Inverfe.

In thefe queftions, there is always given an odd number of terms, either five, or feven, or nine, &c. Thefe are diftinguifhed into terms of fuppofition, and terms of demand, there being always one term more of the former than of the latter, which is of the fame kind with the anfwer fought.

### R U L E.

Set down in the middle place that term of fuppofition which is of the fame kind with the anfwer fought.—Take one of the other terms of fuppofition, and one of the demanding terms which is of the fame kind with it : then place one of them for a firft term, and the other for a third, according to the directions given in the Rule of Three.—Do the fame with another term of fuppofition, and its correfponding demanding term ; and fo on if there be more terms of each kind ; fetting the numbers under each other which fall all on the left-hand fide of the middle term, and the fame for the others on the right-hand fide.—Then, to work

*By feveral Operations.*—Take the two upper terms and the middle term, in the fame order as they ftand, for the firft Rule of Three queftion to be worked, whence will be found a fourth term. Then take this fourth number, fo found, for thé middle term of a fecond Rule of Three queftion, and the next two under terms in the general ftating, in the fame order as they ftand, finding a fourth term for them. And fo on, as far as there are any numbers in the general ftating, making always the fourth number refulting from each fimple ftating to be the fecond term of the next following one. So fhall the laft refulting number be the anfwer to the queftion.

*By One Operation.*—Multiply together all the terms ftanding under each other, on the left-hand fide of the middle term ; and in like manner, multiply together all thofe on the right-hand fide of it. Then multiply the middle term by the latter product, and divide the refult by the former product, fo fhall the quotient be the anfwer fought.

### EXAMPLES.

1. How many men can complete a trench of 135 yards long in 8 days, when 16 men can dig 54 yards in 6 days?

*General Stating.*

yds  54 : 16 men :: 135 yds
days  8                    6 days
     ___                  ___
     432                  810
                          16
                         ___
                         4860
                         81    men
                   432 ) 12960 ( 30 Anf. by one operat.
                         1296
                         ___
                           o

*The same by two operations.*

|  1ft. | 2d. |
|---|---|
| As 54 : 16 :: 135 : 40 | As 8 : 40 :: 6 : 30 |
| 16 | 6 |
| ___ | ___ |
| 810 | 8 ) 240 ( 30 Anf. |
| 135 | 24 |
| 54 ) 2160 ( 40 | ___ |
| 216 | o |
| ___ | |
| o | |

2. If 100l in one year gain 5l intereft, what will be the intereft of 750l for 7 years?　　　　　Anf. 262l 10s

3. If a family of 9 perfons expend 120l in 8 months; how much will ferve a family of 24 people 16 months?　　　　　Anf. 640l

4. If 27s be the wages of 4 men for 7 days; what will be the wages of 14 men for 10 days?　　　　　Anf. 6l 15s

5. If a footman travel 130 miles in 3 days, when the days are 12 hours long; in how many days, of 10 hours each, may he travel 360 miles?　　　　　Anf. $9\frac{6}{5}\frac{3}{5}$ days.

　　　　　Ex. 6.

Ex. 6. If 120 bufhels of corn can ferve 14 horfes 56 days; how many days will 94 bufhels ferve 6 horfes ?

<div align="right">Anf. 102$\frac{16}{45}$ days.</div>

7. If 3000lb of beef ferve 340 men 15 days; how many lbs will ferve 120 men for 25 days ? Anf. 1764lb 11$\frac{15}{51}$oz.

8. If a barrel of beer be fufficient to laft a family of 7 perfons 12 days; how many barrels will be drank by 14 perfons in the fpace of a year ? Anf. 60$\frac{5}{6}$ barrels.

9. If 240 men, in 5 days, of 11 hours each, can dig a trench 230 yards long, 3 wide, and 2 deep; in how many days of 9 hours long, will 24 men dig a trench of 420 yards long, 5 wide, and 3 deep ? Anf. 288$\frac{59}{207}$ days.

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# OF VULGAR FRACTIONS.

A FRACTION, or broken number, is an expreffion of a part, or fome parts, of fomething confidered as a whole.

It is denoted by two numbers, placed one below the other, with a line betwen them :

Thus, $\frac{3}{4}$ numerator } which is named 3-fourths.
denominator

The Denominator, or number placed below the line, fhews how many equal parts the whole quantity is divided into; and reprefents the Divifor in Divifion.—And the Numerator, or number fet above the line, fhews how many of thefe parts are expreffed by the Fraction: being the remainder after divifion.—Alfo, both thefe numbers are, in general, named the Terms of the Fraction.

Fractions are either Proper, Improper, Simple, Compound, or Mixed.

A Proper Fraction, is when the numerator is lefs than the denominator; as $\frac{1}{2}$, or $\frac{2}{3}$, or $\frac{3}{5}$, &c.

An Improper Fraction, is when the numerator is equal to, or exceeds, the denominator; as, $\frac{3}{3}$, or $\frac{4}{4}$, or $\frac{7}{5}$, &c.

A Simple Fraction, is a fingle expreffion, denoting any number of parts of the integer; as $\frac{2}{3}$, or $\frac{3}{2}$.

A Compound Fraction, is the fraction of a fraction, or feveral fractions connected with the word *of* between them; as $\frac{1}{2}$ of $\frac{2}{3}$, or $\frac{3}{5}$ of $\frac{4}{6}$ of 3, &c.

A Mixed Number, is compofed of a whole number and a fraction together; as $3\frac{1}{4}$, or $12\frac{4}{5}$, &c.

<div align="center">E 2</div>

<div align="right">A whole</div>

A whole or integer number may be expreſſed like a fraction, by writing 1 below it, as a denominator; ſo 3 is $\frac{3}{1}$, or 4 is $\frac{4}{1}$, &c.

A fraction denotes diviſion; and its value is equal to the quotient obtained by dividing the numerator by the denominator; ſo $\frac{12}{4}$ is equal to 3, and $\frac{20}{5}$ is equal to 4.

Hence then, if the numerator be leſs than the denominator, the value of the fraction is leſs than 1. If the numerator be the ſame as the denominator, the fraction is juſt equal to 1. And if the numerator be greater than the denominator, the fraction is greater than 1.

---

# REDUCTION of VULGAR FRACTIONS.

REDUCTION of Vulgar Fractions, is the bringing them out of one form or denomination into another; commonly to prepare them for the operations of Addition, Subtraction, &c. of which there are ſeveral caſes.

## PROBLEM.

*To find the Greateſt Common Meaſure of Two or more Numbers.*

THE Common Meaſure of two or more numbers, is that number which will divide them both without remainder; ſo, 3 is a common meaſure of 18 and 24; the quotient of the former being 6, and of the latter 8. And the greateſt number that will do this, is the greateſt common meaſure: ſo 6 is the greateſt common meaſure of 18 and 24; the quotient of the former being 3, and of the latter 4, which will not both divide farther.

## RULE.

IF there be two numbers only; divide the greater by the leſs; then divide the diviſor by the remainder; and ſo on, dividing always the laſt diviſor by the laſt remainder, till nothing remains; ſo ſhall the laſt diviſor of all be the greateſt common meaſure ſought.

When there are more than two numbers; find the greateſt common meaſure of two of them, as before; then do the ſame for that common meaſure and another of the numbers;

and

and fo on, through all the numbers; fo will the greateft common meafure laft found be the anfwer.

If it happen that the common meafure thus found is 1 ; then the numbers are faid to be incommenfurable, or not having any common meafure.

## EXAMPLES.

1. To find the greateft common meafure of 1998, 918, and 522.

918 ) 1998 ( 2            So, 54 is the greateft common
    1836                  meafure of 1998 and 918.
    162 ) 918 ( 5    Hence 54 ) 522 ( 9
      810                       486
      108 ) 162 ( 1        36 ) 54 ( 1
       108                   36
       54 ) 108 ( 2      18 ) 36 ( 2
       108                 36

So that 18 is the anfwer required.

2. What is the greateft common meafure of 246 and 372 ?
Anf. 6.

3. What is the greateft common meafure of 336, 720, and 1736 ?
Anf. 8.

## CASE 1.

*To Abbreviate or Reduce Fractions to their Loweft Terms.*

### RULE*.

DIVIDE the terms of the given fraction by any number that will divide them without a remainder ; then divide thefe quotients

---

* That dividing both the terms of the fraction by the fame number, whatever it be, will give another fraction equal to the former, is evident. And when thefe divifions are performed as often as can be done, or when the common divifor is the greateft poffible, the terms of the refulting fraction muft be the leaft poffible.

*Note.* 1. Any number ending with an even number, or a cipher, is divifible, or can be divided, by 2.

2. Any number ending with 5, or 0, is divifible by 5.

3. If

quotients again in the fame manner; and fo on, till it appears that there is no number greater than 1 which will divide them; then the fraction will be in its loweft terms.

*Or*, Divide both the terms of the fraction by their greateft common meafure at once, and the quotients will be the terms of the fraction required, of the fame value as at firft.

## EXAMPLES.

1. Reduce $\frac{144}{240}$ to its leaft terms.

$$\frac{144}{240} = \frac{72}{120} = \frac{36}{60} = \frac{18}{30} = \frac{9}{15} = \frac{3}{5}, \text{ the Anfwer.}$$

Or thus:

144 ) 240 ( 1
    144
   96 ) 144 ( 1
     96
      48 ) 96 ( 2
       96

Therefore 48 is the greateft common meafure; and 48 ) $\frac{144}{240} = \frac{3}{5}$ the Anfwer, the fame as before.

2. Reduce

---

3. If the right-hand place of any number be 0, the whole is divifible by 10; if there be two ciphers, it is divifible by 100; if three ciphers, by 1000: and fo on; which is only cutting off thofe ciphers.

4. If the two right-hand figures of any number be divifible by 4, the whole is divifible by 4. And if the three right-hand figures be divifible by 8, the whole is divifible by 8. And fo on.

5. If the fum of the digits in any number be divifible by 3, or by 9, the whole is divifible by 3, or by 9.

6. If the right-hand digit be even, and the fum of all the digits be divifible by 6, then the whole will be divifible by 6.

7. A number is divifible by 11, when the fum of the 1ft, 3d, 5th, &c. or all the odd places, is equal to the fum of the 2d, 4th, 6th, &c. or of all the even places of digits.

8. If a number cannot be divided by fome quantity lefs than the fquare of the fame, that number is a prime, or cannot be divided by any number whatever.

9. All prime numbers, except 2 and 5, have either 1, 3, 7, or 9, in the place of units; and all other numbers are compofite, or can be divided.

10. When

2. Reduce $\frac{192}{576}$ to its loweft terms.     Anf. $\frac{1}{3}$.

3. Reduce $\frac{252}{304}$ to its loweft terms.     Anf. $\frac{9}{13}$.

4. Reduce $\frac{1344}{1536}$ to its loweft terms.     Anf. $\frac{7}{8}$.

### CASE II.

*To Reduce a Mixed Number to its Equivalent Improper Fraction.*

### RULE*.

MULTIPLY the whole number by the denominator of the fraction, and add the numerator to the product; then fet that fum above the denominator for the fraction required.

### EXAMPLES.

1. Reduce $23\frac{2}{5}$ to a fraction.

$$
\begin{array}{r}
23 \\
5 \\
\hline
115 \\
2 \\
\hline
117 \\
5
\end{array}
$$

Or,

$$\frac{(23 \times 5) + 2}{5} = \frac{117}{5}, \text{ the Anfwer.}$$

2. Reduce $12\frac{7}{9}$ to a fraction.     Anf. $\frac{115}{9}$.

3. Reduce $14\frac{7}{10}$ to a fraction.     Anf. $\frac{147}{10}$.

4. Reduce $183\frac{5}{21}$ to a fraction.     Anf. $\frac{3848}{21}$.

---

10. When numbers, with the fign of addition or fubtraction between them, are to be divided by any number, then each of thofe numbers muft be divided by it. Thus $\dfrac{10 + 8 - 4}{2} = 5 + 4 - 2 = 7.$

11. But if the numbers have the fign of multiplication between them, only one of them muft be divided. Thus,

$$\frac{10 \times 8 \times 3}{6 \times 2} = \frac{10 \times 4 \times 3}{6 \times 1} = \frac{10 \times 4 \times 1}{2 \times 1} = \frac{10 \times 2 \times 1}{1 \times 1} = \frac{20}{1} = 20.$$

* This is no more than firft multiplying a quantity by fome number, and then dividing the refult back again by the fame, which it is evident does not alter the value; for any fraction reprefents a divifion of the numerator by the denominator.

## CASE III.

*To Reduce an Improper Fraction to its Equivalent Whole or Mixed Number.*

### RULE*.

DIVIDE the numerator by the denominator, and the quotient will be the whole or mixed number fought.

### EXAMPLES.

1. Reduce $\frac{12}{3}$ to its equivalent number.

Here $\frac{12}{3}$ or $12 \div 3 = 4$, the Anfwer.

2. Reduce $\frac{15}{7}$ to its equivalent number.

Here $\frac{15}{7}$ or $15 \div 7 = 2\frac{1}{7}$, the Anfwer.

3. Reduce $\frac{749}{17}$ to its equivalent number.

Thus, $17 ) 749 ( 44\frac{1}{17}$
$\qquad \underline{68}$
$\qquad \quad 69 \qquad$ So that $\frac{749}{17} = 44\frac{1}{17}$, the Anfwer.
$\qquad \quad \underline{68}$
$\qquad \quad \overline{\phantom{xx}1}$

4. Reduce $\frac{56}{7}$ to its equivalent number.     Anf. 8.
5. Reduce $\frac{1362}{25}$ to its equivalent number.    Anf. $54\frac{12}{25}$.
6. Reduce $\frac{2918}{17}$ to its equivalent number.    Anf. $171\frac{11}{17}$.

## CASE IV.

*To Reduce a Whole Number to an Equivalent Fraction, having a Given Denominator.*

### RULE†.

MULTIPLY the whole number by the given denominator; then fet the product over the faid denominator, and it will form the fraction required.

_____

* This Rule is evidently the reverfe of the former; and the reafon of it is manifeft from the nature of Common Divifion.

† Multiplication and Divifion being here equally ufed, the refult muft be the fame as the quantity firft propofed.

               EXAM-

## EXAMPLES.

1. Reduce 9 to a fraction whofe denominator fhall be 7.

Here $9 \times 7 = 63$, then $\frac{63}{7}$ is the Anfwer.

For $\frac{63}{7} = 63 \div 7 = 9$, the Proof.

2. Reduce 13 to a fraction whofe denominator fhall be 12.

Anf. $\frac{156}{12}$.

3. Reduce 27 to a fraction whofe denominator fhall be 11.

Anf. $\frac{297}{11}$.

## CASE V.

*To Reduce a Compound Fraction to an Equivalent Simple One.*

### RULE*.

MULTIPLY all the numerators together for a numerator, and all the denominators together for a denominator, and they will form the fimple fraction fought.

When part of the compound fraction is a whole or mixed number, it muft firft be reduced to a fraction by one of the former cafes.

And, when it can be done, any two terms of the fraction may be divided by the fame number, and the quotients ufed inftead of them. Or when there are terms that are common, they may be omitted.

## EXAMPLES.

1. Reduce $\frac{1}{2}$ of $\frac{2}{3}$ of $\frac{3}{4}$ to a fimple fraction.

Here $\frac{1 \times 2 \times 3}{2 \times 3 \times 4} = \frac{6}{24} = \frac{1}{4}$, the Anfwer.

Or, $\frac{1 \times 2 \times 3}{2 \times 3 \times 4} = \frac{1}{4}$, by omitting the 2's and 3's.

---

* The truth of this Rule may be fhewn as follows : Let the compound fraction be $\frac{2}{3}$ of $\frac{5}{7}$. Now $\frac{1}{3}$ of $\frac{5}{7}$ is $\frac{5}{7} \div 3$, which is $\frac{5}{21}$; confequently $\frac{2}{3}$ of $\frac{5}{7}$ will be $\frac{5}{21} \times 2$ or $\frac{10}{21}$; that is, the numerators are multiplied together, and alfo the denominators, as in the Rule.—When the compound fraction confifts of more than two fingle ones ; having firft reduced two of them as above, then the refulting fraction and a third will be the fame as a compound fraction of two parts; and fo on to the laft of all.

2. Reduce

2. Reduce $\frac{2}{3}$ of $\frac{3}{5}$ of $\frac{10}{11}$ to a fimple fraction.

Here $\dfrac{2 \times 3 \times 10}{3 \times 5 \times 11} = \dfrac{60}{165} = \dfrac{12}{33} = \dfrac{4}{11}$, the Anfwer.

Or $\dfrac{2 \times 3 \times 10}{3 \times 5 \times 11} = \dfrac{4}{11}$, the fame as before.

3. Reduce $\frac{3}{7}$ of $\frac{4}{5}$ to a fimple fraction.    Anf. $\frac{12}{35}$.

4. Reduce $\frac{2}{3}$ of $\frac{3}{5}$ of $\frac{5}{8}$ to a fimple fraction.    Anf. $\frac{1}{4}$.

5. Reduce $\frac{2}{5}$ of $\frac{5}{8}$ of $3\frac{1}{2}$ to a fimple fraction.    Anf. $\frac{7}{8}$.

6. Reduce $\frac{2}{7}$ of $\frac{5}{8}$ of $\frac{7}{2}$ of 4 to a fimple fraction.  Anf. $\frac{5}{2}$.

7. Reduce $2\frac{2}{5}$ of $\frac{5}{6}$ to a fraction.    Anf. $\frac{7}{3}$.

### CASE VI.

*To Reduce Fractions of Different Denominators to Equivalent Fractions, having a Common Denominator.*

### RULE*.

MULTIPLY each numerator by all the denominators except its own, for the new numerators; and multiply all the denominators together for a common denominator.

*Note*, It is evident, that in this and feveral other operations, when any of the propofed quantities are integers, or mixed numbers, or compound fractions, they muft be reduced, by their proper Rules, to the form of fimple fractions.

### EXAMPLES.

1. Reduce $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$ to a common denominator.

1 × 3 × 4 = 12 the new numerator for $\frac{1}{2}$.

2 × 2 × 4 = 16    -    -    ditto for $\frac{2}{3}$.

3 × 2 × 3 = 18    -    -    ditto for $\frac{3}{4}$.

2 × 3 × 4 = 24 the common denominator.

Therefore the equivalent fractions are $\frac{12}{24}$, $\frac{16}{24}$, and $\frac{18}{24}$.

Or the whole operation of multiplying may be very well performed mentally, only fetting down the refults and

---

* This is evidently no more than multiplying each numerator and its denominator by the fame quantity; and confequently the value of the fraction is not altered.

given

given fractions thus: $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$ = $\frac{12}{24}$, $\frac{16}{24}$, $\frac{18}{24}$ = $\frac{6}{12}$, $\frac{8}{12}$, $\frac{9}{12}$, by abbreviation.

2. Reduce $\frac{2}{7}$ and $\frac{5}{9}$ to fractions of a common denominator.

Anſ. $\frac{18}{63}$, $\frac{35}{63}$.

3. Reduce $\frac{2}{3}$, $\frac{3}{5}$, and $\frac{3}{4}$ to a common denominator.

Anſ. $\frac{40}{60}$, $\frac{36}{60}$, $\frac{45}{60}$.

4. Reduce $\frac{5}{6}$, $2\frac{3}{5}$, and 4 to a common denominator.

Anſ. $\frac{25}{30}$, $\frac{78}{30}$, $\frac{120}{30}$.

*Note* 1. When the denominators of two given fractions have a common meaſure, let them be divided by it ; then multiply the terms of each given fraction by the quotient ariſing from the other's denominator.

2. When the leſs denominator of two fractions exactly divides the greater, multiply the terms of that which hath the leſs denominator by the quotient.

3. When more than two fractions are propoſed ; it is ſometimes convenient, firſt to reduce two of them to a common denominator ; then theſe and a third ; and ſo on till they be all reduced to their leaſt common denominator.

## CASE VII.

### *To find the Value of a Fraction in Parts of the Integer.*

#### RULE.

MULTIPLY the integer by the numerator, and divide the product by the denominator, by Compound Multiplication and Diviſion, if the integer be a compound quantity.

Or, if it be a ſingle integer, multiply the numerator by the parts in the next inferior denomination, and divide the product by the denominator. Then, if any thing remains, multiply it by the parts in the next inferior denomination, and divide by the denominator as before ; and ſo on as far as neceſſary ; ſo ſhall the quotients, placed in order, be the value of the fraction required*.

---

* The numerator of a fraction being conſidered as a remainder, in Diviſion, and the denominator as the diviſor, this rule is of the ſame nature as Compound Diviſion, or the valuation of remainders in the Rule of Three, before explained.

### EXAMPLES.

1. What is the $\frac{4}{5}$ of 2l 6s?

By the former part of the Rule,

```
        2l   6s
         4
      ———————
     5 ) 9   4
Anf.    1l 16s 9d 2⅖q.
```

2. What is the value of $\frac{2}{3}$ of 1l?

By the 2d part of the Rule,

```
        2.
       20
     ——————
  3 ) 40 ( 13s 4d Anf.
     ——————
        1
      · 12
     ——————
  3 ) 12 ( 4d
```

3. Find the value of $\frac{3}{8}$ of a pound sterling?    Anf. 7s 6d.
4. What is the value of $\frac{2}{9}$ of a guinea?       Anf. 4s 8d.
5. What is the value of $\frac{3}{4}$ of a half crown? Anf. 1s 10½d.
6. What is the value of $\frac{2}{5}$ of 4s 10d?      Anf. 1s 11⅕d.
7. What is the value of $\frac{3}{5}$lb. troy?     Anf. 7oz 4dwts.
8. What is the value of $\frac{5}{16}$ of a cwt?     Anf. 1qr 7lb.
9. What is the value of $\frac{5}{8}$ of an acre?    Anf. 2ro 20po.
10. What is the value of $\frac{3}{10}$ of a day?    Anf. 7hrs 12min.

### CASE VIII.

*To Reduce a Fraction from one Denomination to another.*

### RULE*.

Consider how many of the lefs denomination make one of the greater ; then multiply the numerator by that number, if the reduction be to a lefs name, or multiply the denominator, if to a greater.

### EXAMPLES.

1. Reduce $\frac{2}{9}$ of a pound to the fraction of a penny.

$$\frac{2}{9} \times \frac{20}{1} \times \frac{12}{1} = \frac{480}{9} = \frac{160}{3}, \text{ the Anfwer.}$$

---

* This is the fame as the Rule of Reduction in whole numbers, from one denomination to another.

2. Reduce $\frac{5}{6}$ of a penny to the fraction of a pound.

$$\frac{5}{6} \times \frac{1}{12} \times \frac{1}{20} = \frac{5}{1440} = \frac{1}{288}, \text{ the Anfwer.}$$

3. Reduce $\frac{2}{15}$l to the fraction of a penny.  Anf. $\frac{32}{1}$d.
4. Reduce $\frac{2}{3}$q to the fraction of a pound.  Anf. $\frac{1}{1440}$.
5. Reduce $\frac{2}{7}$cwt to the fraction of a lb.  Anf. $\frac{32}{1}$.
6. Reduce $\frac{4}{5}$dwt to the fraction of a lb troy.  Anf. $\frac{1}{300}$.
7. Reduce $\frac{5}{8}$crown to the fract. of a guinea.  Anf. $\frac{25}{168}$.
8. Reduce $\frac{5}{6}$half-crown to the fract. of a fhilling.  Anf. $\frac{25}{12}$.
9. Reduce 2s 6d to the fraction of a £.  Anf. $\frac{1}{8}$.
10. Reduce 17s 7d $3\frac{3}{5}$q to the fraction of a £.

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# ADDITION of VULGAR FRACTIONS.

### RULE.

If the fractions have a common denominator; add all the numerators together, then place the fum over the common denominator, and that will be the fum of the fractions required.

If the propofed fractions have not a common denominator; they muft be reduced to one. Alfo compound fractions muft be reduced to fimple ones; and mixed numbers to improper fractions: alfo fractions of different denominations to thofe of the fame denomination*.

---

* Before fractions are reduced to a common denominator, they are quite diffimilar, as much as fhillings and pence are, and therefore cannot be incorporated with one another, any more than thefe can. But when they are reduced to a common denominator, and made parts of the fame thing, their fum, or difference, may then be as properly expreffed by the fum or difference of the numerators, as the fum or difference of any two quantities whatever, by the fum or difference of their individuals. Whence the reafon of the Rule is manifeft, both for Addition and Subtraction.

When feveral fractions are to be collected, it is commonly beft firft to add two of them together that moft eafily reduce to a common denominator; then add their fum and a third, and fo on.

EXAM-

### EXAMPLES.

1. To add $\frac{3}{5}$ and $\frac{4}{5}$ together.

   Here $\frac{3}{5} + \frac{4}{5} = \frac{7}{5} = 1\frac{2}{5}$, the Answer.

2. To add $\frac{3}{5}$ and $\frac{5}{6}$ together.

   $\frac{3}{5} + \frac{5}{6} = \frac{18}{30} + \frac{25}{30} = \frac{43}{30} = 1\frac{13}{30}$, the Answer.

3. To add $\frac{5}{8}$ and $7\frac{1}{2}$ and $\frac{1}{3}$ of $\frac{3}{4}$ together.

   $\frac{5}{8} + 7\frac{1}{2} + \frac{1}{3}$ of $\frac{3}{4} = \frac{5}{8} + \frac{15}{2} + \frac{1}{4} = \frac{5}{8} + \frac{60}{8} + \frac{2}{8} = \frac{67}{8} = 8\frac{3}{8}$.

4. To add $\frac{3}{7}$ and $\frac{6}{7}$ together.                Anf. $1\frac{2}{7}$.

5. To add $\frac{3}{4}$ and $\frac{5}{9}$ together.                Anf. $1\frac{11}{36}$.

6. Add $\frac{2}{7}$ and $\frac{5}{14}$ together.                Anf. $\frac{9}{14}$.

7. What is the fum of $\frac{2}{3}$ and $\frac{3}{5}$ and $\frac{5}{7}$?         Anf. $1\frac{103}{105}$.

8. What is the fum of $\frac{5}{9}$ and $\frac{3}{5}$ and $2\frac{1}{6}$?        Anf. $3\frac{29}{90}$.

9. What is the fum of $\frac{3}{5}$ and $\frac{4}{5}$ of $\frac{1}{3}$; and $9\frac{3}{20}$?  Anf. $10\frac{1}{60}$.

10. What is the fum of $\frac{2}{3}$ of a pound and $\frac{5}{9}$ of a fhilling?

                                      Anf. $\frac{125}{9}$s or 13s 10d $2\frac{2}{3}$q.

11. What is the fum of $\frac{3}{5}$ of a fhilling and $\frac{4}{15}$ of a penny?

                                      Anf. $\frac{112}{15}$d or 7d $1\frac{13}{15}$q.

12. What is the fum of $\frac{1}{7}$ of a pound, and $\frac{2}{9}$ of a fhilling, and $\frac{5}{12}$ of a penny?         Anf. $\frac{3139}{1008}$s or 3s 1d $1\frac{10}{21}$q.

13. To fum $7\frac{3}{5}$ of $\frac{4}{7} + \frac{3}{5}$ of $\frac{4}{7}$ of $7 + 5\frac{2}{3} + \frac{9}{11}$.

                                      Anf. $16\frac{263}{1155}$.

---

## SUBTRACTION of VULGAR FRACTIONS.

### RULE.

PREPARE the fractions the fame as for Addition, when neceffary; then fubtract the one numerator from the other, and fet the remainder over the common denominator, for the difference of the fractions fought.

### EXAMPLES.

1. To find the difference between $\frac{5}{6}$ and $\frac{1}{6}$.

   Here $\frac{5}{6} - \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$, the Answer.

2. To find the difference between $\frac{3}{4}$ and $\frac{5}{7}$.

   $\frac{3}{4} - \frac{5}{7} = \frac{21}{28} - \frac{20}{28} = \frac{1}{28}$, the Answer.

                                      3. What

3. What is the difference between $\frac{5}{12}$ and $\frac{7}{12}$ ?    Anf. $\frac{1}{6}$.

4. What is the difference between $\frac{3}{13}$ and $\frac{4}{39}$ ?    Anf. $\frac{5}{39}$.

5. What is the difference between $\frac{5}{12}$ and $\frac{7}{13}$ ?    Anf. $\frac{19}{156}$.

6. What is the diff. between $5\frac{3}{8}$ and $\frac{2}{7}$ of $4\frac{1}{6}$ ? Anf. $4\frac{31}{168}$.

7. What is the difference between $\frac{5}{9}$ of a pound, and $\frac{2}{3}$ of $\frac{3}{4}$ of a fhilling ?                Anf. $\frac{191}{18}$s or 10s 7d $1\frac{1}{3}$q.

8. What is the difference between $\frac{2}{7}$ of $5\frac{1}{6}$ of a pound, and $\frac{3}{5}$ of a fhilling ?        Anf. $\frac{30371}{2100}$l or 11l 8s $11\frac{3}{35}$d.

———

# MULTIPLICATION of VULGAR FRACTIONS.

### RULE*.

REDUCE mixed numbers, if there be any, to equivalent fractions; then multiply all the numerators together for a numerator, and all the denominators together for a denominator, which will give the product required.

### EXAMPLES.

1. Required the product of $\frac{3}{4}$ and $\frac{2}{9}$.

Here $\frac{3}{4} \times \frac{2}{9} = \frac{6}{36} = \frac{1}{6}$, the Anfwer.

Or $\frac{3}{4} \times \frac{2}{9} = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$.

2. Required the continued product of $\frac{2}{3}$, $3\frac{1}{4}$, 5, and $\frac{3}{4}$ of $\frac{3}{5}$.

Here $\frac{2}{3} \times \frac{13}{4} \times \frac{5}{1} \times \frac{3}{4} \times \frac{3}{5} = \frac{13 \times 3}{4 \times 2} = \frac{39}{8}$

$= 4\frac{7}{8}$, Anfwer.

3. Required the product of $\frac{2}{7}$ and $\frac{5}{8}$.        Anf. $\frac{5}{28}$.

4. Required the product of $\frac{4}{15}$ and $\frac{5}{24}$.        Anf. $\frac{1}{18}$.

———

* Multiplication of any thing by a fraction implies the taking fome part or parts of the thing; it may therefore be truly expreffed by a compound fraction; which is refolved by multiplying together the numerators and the denominators.

*Note*, A fraction is beft multiplied by an integer, by dividing the denominator by it; but if it will not exactly divide, then multiply the numerator by it.

5. Re-

5. Required the product of $\frac{3}{7}$, $\frac{4}{9}$, and $\frac{14}{15}$.     Anf. $\frac{8}{45}$.

6. Required the product of $\frac{1}{2}$, $\frac{2}{3}$, and 3.     Anf. 1.

7. Required the product of $\frac{7}{9}$, $\frac{3}{5}$, and $4\frac{5}{14}$.     Anf. $2\frac{1}{30}$.

8. Required the product of $\frac{5}{6}$, and $\frac{2}{3}$ of $\frac{6}{7}$.     Anf. $\frac{10}{21}$.

9. Required the product of 6, and $\frac{2}{3}$ of 5.     Anf. 20.

10. Required the product of $\frac{2}{9}$ of $\frac{3}{5}$, and $\frac{5}{8}$ of $3\frac{2}{7}$.     Anf. $\frac{2/3}{84}$.

11. Required the product of $3\frac{2}{7}$, and $4\frac{14}{33}$.     Anf. $14\frac{124}{231}$.

12. Required the product of 5, $\frac{2}{3}$, $\frac{2}{7}$ of $\frac{3}{5}$, and $4\frac{1}{6}$.     Anf. $2\frac{8}{21}$.

---

# DIVISION of VULGAR FRACTIONS.

### RULE*.

PREPARE the fractions as before in Multiplication; then divide the numerator by the numerator, and the denominator by the denominator, if they will exactly divide; but if not, then invert the terms of the divifor, and multiply the dividend by it, as in Multiplication.

### EXAMPLES.

1. Divide $\frac{25}{9}$ by $\frac{5}{3}$.

Here $\frac{25}{9} \div \frac{5}{3} = \frac{5}{3} = 1\frac{2}{3}$, by the firft method.

2. Divide $\frac{5}{9}$ by $\frac{2}{15}$.

Here $\frac{5}{9} \div \frac{2}{15} = \frac{5}{9} \times \frac{15}{2} = \frac{5}{3} \times \frac{5}{2} = \frac{25}{6} = 4\frac{1}{6}$.

3. It is required to divide $\frac{16}{25}$ by $\frac{4}{5}$.     Anf. $\frac{4}{5}$.

4. It is required to divide $\frac{7}{16}$ by $\frac{3}{4}$.     Anf. $\frac{7}{12}$.

5. It is required to divide $\frac{14}{9}$ by $\frac{7}{6}$.     Anf. $1\frac{1}{3}$.

6. It is required to divide $\frac{5}{6}$ by $\frac{15}{7}$.     Anf. $\frac{7}{18}$.

---

* Divifion being the reverfe of Multiplication, the reafon of the Rule is evident.

*Note,* A fraction is beft divided by an integer, by dividing the numerator by it; but if it will not exactly divide, then multiply the denominator by it.

7. It is required to divide $\frac{12}{35}$ by $\frac{3}{5}$.        Anf. $\frac{4}{7}$.

8. It is required to divide $\frac{2}{7}$ by $\frac{3}{5}$.        Anf. $\frac{10}{21}$.

9. It is required to divide $\frac{9}{16}$ by 3.        Anf. $\frac{3}{16}$.

10. It is required to divide $\frac{3}{5}$ by 2.        Anf. $\frac{3}{10}$.

11. It is required to divide $7\frac{1}{3}$ by $9\frac{5}{9}$.        Anf. $\frac{33}{43}$.

12. It is required to divide $\frac{2}{3}$ of $\frac{1}{3}$ by $\frac{5}{7}$ of $7\frac{3}{5}$.    Anf. $\frac{7}{171}$.

---

# RULE-OF-THREE in VULGAR FRACTIONS.

### RULE.

MAKE the neceſſary preparations as before directed; then multiply continually together, the ſecond and third terms, and the firſt with its parts inverted as in Diviſion, for the anſwer *.

### EXAMPLES.

1. If $\frac{3}{8}$ of a yard of velvet coſt $\frac{2}{5}$ of a pound ſterling; what will $\frac{5}{16}$ of a yard coſt?

$$\frac{3}{8} : \frac{2}{5} :: \frac{5}{16} : \frac{\cancel{8}}{3} \times \frac{\cancel{2}}{\cancel{5}} \times \frac{\cancel{5}}{\cancel{16}} = \frac{1}{3}\,l = 6s\ 8d. \text{ Anſwer.}$$

2. What will $3\frac{3}{8}$oz of ſilver coſt, at 6s 4d an ounce?      Anſ. $1l$ 1s $4\frac{1}{2}$d.

3. If $\frac{3}{16}$ of a ſhip be worth 273l 2s 6d, what is $\frac{5}{32}$ of her worth?      Anſ. 227l 12s 1d.

4. What is the purchaſe of 1230l bank-ſtock, at $108\frac{5}{8}$ per cent?      Anſ. 1336l 1s 9d.

5. What is the intereſt of 273l 15s for a year, at $3\frac{1}{4}$ per cent?      Anſ. 8l 17s $11\frac{1}{4}$d.

6. If $\frac{1}{8}$ of a ſhip be worth 73l 1s 3d, what part of her is worth 250l 10s?      Anſ. $\frac{3}{7}$.

---

* This is only multiplying the 2d and 3d terms together, and dividing the product by the firſt, as in the Rule-of-Three in whole numbers.

7. What length muſt be cut off a board, that is $7\frac{1}{4}$ inches broad, to contain a ſquare foot, or as much as another piece of 12 inches long and 12 broad ?          Anſ. $18\frac{18}{31}$ inches.

8. What quantity of ſhalloon, that is $\frac{3}{4}$ of a yard wide, will line $9\frac{1}{2}$ yards of cloth that is $2\frac{1}{2}$ yards wide ?     Anſ. $31\frac{2}{3}$ yds.

9. If the penny-loaf weigh $6\frac{9}{10}$ oz when the price of wheat is 5s the buſhel ; what ought it to weigh when the wheat is at 8s 6d the buſhel ?          Anſ. $4\frac{1}{17}$ oz.

10. How much in length, of a piece of land that is $11\frac{1}{2}$ poles broad, will make an acre of land, or as much as 40 poles in length and 4 in breadth ?          Anſ. $13\frac{61}{143}$ poles.

11. If a courier perform a certain journey in $35\frac{1}{2}$ days, travelling $13\frac{5}{8}$ hours a-day ; how long would he be in performing the ſame, travelling only $11\frac{9}{10}$ hours a-day ?

Anſ. $40\frac{615}{952}$ days.

12. A regiment of ſoldiers, conſiſting of 976 men, are to be new clothed, each coat to contain $2\frac{1}{2}$ yards of cloth that is $1\frac{5}{8}$ yard wide, and lined with ſhalloon $\frac{7}{8}$ yard wide; how many yards of ſhalloon will line them ?

Anſ. 4531 yds 1 qr $2\frac{6}{7}$ nails.

---

## DECIMAL FRACTIONS.

A Decimal Fraction is that which has for its denominator, an unit (1) with as many ciphers annexed as the numerator has places ; and it is uſually expreſſed by ſetting down the numerator only, with a point before it, on the left-hand. Thus, $\frac{5}{10}$ is ·5, and $\frac{25}{100}$ is ·25, and, $\frac{75}{1000}$ is ·075, and $\frac{124}{100000}$ is ·00124 ; where ciphers are prefixed to make up as many places as are ciphers in the denominator, when there is a deficiency of figures.

A mixed number is made up of a whole number with ſome decimal fraction, the one being ſeparated from the other by a point. Thus 3·25 is the ſame as $3\frac{25}{100}$. or $\frac{325}{100}$.

Ciphers on the right-hand of decimals make no alteration in their value ; for ·5 or ·50 or ·500, are decimals having all the ſame value, being each $= \frac{5}{10}$ or $\frac{1}{2}$. But if they are placed on the left-hand, they decreaſe the value in a tenfold proportion : Thus ·5 is $\frac{5}{10}$ or 5 tenths, but ·05 is only $\frac{5}{100}$ or 5 hundreds, and ·005 is but $\frac{5}{1000}$ or 5 thouſandths.

The

The 1st place of decimals, counted from the left-hand towards the right, is called the place of primes, or 10ths; the 2d is the place of feconds, or 100ths; the 3d is the place of thirds, or 1000ths; and fo on. For, in decimals, as well as in whole numbers, the values of the places increafe towards the left-hand, and decreafe towards the right, both in the fame tenfold proportion; as in the following Scale or Table of Notation:

| millions | hundred thoufands | ten thoufands | thoufands | hundreds | tens | units | tenth parts | hundredth parts | thoufandth parts | ten thoufandth parts | hundred thoufandth parts | millionth parts |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |

## ADDITION of DECIMALS.

### RULE.

Set the numbers under each other according to the value of their places, like as in whole numbers; in which ftate the decimal feparating points will ftand all exactly under each other. Then, beginning at the right hand, add up all the columns of numbers as in integers; and point off as many places, for decimals, as are in the greateft number of decimal places in any of the lines that are added; or place the point directly below all the other points.

### EXAMPLES.

1. To add together 29·0146, and 3146·5, and 2109, and ·62417, and 14·16

$$
\begin{array}{r}
29\cdot0146 \\
3146\cdot5 \\
2109\cdot \\
\cdot62417 \\
14\cdot16 \\
\hline
5299\cdot29877 \text{ the Sum.} \\
\hline
\end{array}
$$

Ex. 2.

Ex. 2. To find the fum of $376 \cdot 25 + 86 \cdot 125 + 637 \cdot 4725$ $+ 6 \cdot 5 + 41 \cdot 02 + 358 \cdot 865$.      Anf. $1506 \cdot 2325$.

3. Required the fum of $3 \cdot 5 + 47 \cdot 25 + 2 \cdot 0073 + 927 \cdot 01$ $+ 1 \cdot 5$.        Anf. $981 \cdot 2673$.

4. Required the fum of $276 + 54 \cdot 321 + 112 + 0 \cdot 65 +$ $12 \cdot 5 + \cdot 0463$.       Anf. $455 \cdot 5173$.

## SUBTRACTION of DECIMALS.

### RULE.

Place the numbers under each other according to the value of their places, as in the laſt Rule. Then, beginning at the right-hand, fubtraƐt as in whole numbers, and point off the decimals as in Addition.

### EXAMPLES.

1. To find the difference between $91 \cdot 73$ and $2 \cdot 138$.

$$91 \cdot 73$$
$$2 \cdot 138$$
<hr>

Anf. $89 \cdot 592$ the Difference.

2. Find the diff. between $1 \cdot 9185$ and $2 \cdot 73$.   Anf. $0 \cdot 8115$.

3. To fubtraƐt $4 \cdot 90142$ from $214 \cdot 81$.     Anf. $209 \cdot 90858$.

4. Find the diff. between $2714$ and $\cdot 916$.   Anf. $2713 \cdot 084$.

## MULTIPLICATION of DECIMALS.

### RULE*.

Place the faƐtors, and multiply them together the fame as if they were whole numbers.—Then point off in the pro-

---

* The Rule will be evident from this example: Let it be required to multiply $\cdot 12$ by $\cdot 361$; thefe numbers are equivalent to $\frac{12}{100}$ and $\frac{361}{1000}$; the product of which is $\frac{4332}{100000} = \cdot 04332$, by the nature of Notation, which confiſts of as many places as there are ciphers, that is, of as many places as there are in both numbers. And in like manner for any other numbers,

                                      duƐt

duct juft as many places of decimals as there are decimals in both the factors. But if there be not fo many figures in the product, then fupply the defect by prefixing ciphers.

### EXAMPLES.

1. Multiply ·321096
by    ·2465

$$\begin{array}{r} 1605480 \\ 1926576 \\ 1284384 \\ 642192 \end{array}$$

Anf. ·0791501640 the Product.

2. Multiply 79·347 by 23·15.       Anf. 1836·88305.
3. Multiply ·63478 by ·8204.       Anf. ·520773512.
4. Multiply ·385746 by ·00464.      Anf. ·00178986144.

### CONTRACTION. I.

*To multiply Decimals by* 1 *with any number of Ciphers, as by* 10, *or* 100, *or* 1000, &c.

THIS is done by only removing the decimal point fo many places farther to the right-hand as there are ciphers in the multiplier ; and fubjoining ciphers if need be.

### EXAMPLES.

1. The product of 51·3 and 1000 is 51300.
2. The product of 2·714 and 100 is
3. The product of ·916 and 1000 is
4. The product of 21·31 and 10000 is

### CONTRACTION II.

*To Contract the Operation, fo as to retain only as many Decimals in the Product as may be thought Neceffary, when the Product would naturally contain feveral more Places.*

SET the units place of the multiplier under that figure of the multiplicand whofe place is the fame as is to be retained for the laft in the product ; and difpofe of the reft of the
figures

figures in the inverted or contrary order to what they are ufually placed in.—Then, in multiplying, reject all the figures that are more to the right than each multiplying figure, and fet down the products, fo that their right-hand figures may fall in a column ftraight below each other; but obferving to increafe the firft figure of every line with what would arife from the figures omitted, in this manner, namely 1 from 5 to 14, 2 from 15 to 24, 3 from 25 to 34, &c; and the fum of all the lines will be the product as required, commonly to the neareft unit in the laft figure.

<div align="center">E X A M P L E S.</div>

1. To multiply 27·14986 by 92·41035, fo as to retain only four places of decimals in the product.

| Contracted Way. | Common Way. |
|---|---|
| 27·14986 | 27·14986 |
| 53014·29 | 92·41035 |
| | |
| 24434874 | 13\|574930 |
| 542997 | 81\|44958 |
| 108599 | 2714\|986 |
| 2715 | 108599\|44 |
| 81 | 542997\|2 |
| 14 | 24434874\| |
| | |
| 2508·9280 | 2508·9280\|650510 |

2. Multiply 480·14936 by 2·72416, retaining only four decimals in the product.

3. Multiply 2490·3048 by ·573286, retaining only five decimals in the product.

4. Multiply 325·701428 by ·7218393, retaining only three decimals in the product.

# DIVISION of DECIMALS.

### RULE.

DIVIDE as in whole numbers; and point off in the quotient as many places for decimals, as the decimal places in the dividend exceed thofe in the divifor*.

Another way to know the place for the decimal point is this : The firft figure of the quotient muft be made to occupy the fame place, of integers or decimals, as doth that figure of the dividend which ftands over the unit's figure of the firft product.

When the places of the quotient are not fo many as the Rule requires, the defect is to be fupplied by prefixing ciphers.

When there happens to be a remainder after the divifion ; or when the decimal places in the divifor are more than thofe in the dividend ; then ciphers may be annexed to the dividend, and the quotient carried on as far as required.

### EXAMPLES.

```
        1.                          2.
179 ) ·48624097 ( ·00271643 | ·2685 ) 27·00000 ( 100·55865
      1282                            15000
        294                            15750
       1150                            23250
        769                            17700
        537                            15900
        000                            24750
      _____                         _____
```

3. Divide 234·70525 by 64·25.          Anf. 3·653.
4. Divide 14 by ·7854.                 Anf. 17·825.
5. Divide 2175·68 by 100.              Anf. 21·7568.
6. Divide ·8727587 by ·162.            Anf. 5·38739.

### CONTRACTION I.

WHEN the divifor is an integer, with any number of ciphers annexed ; cut off thofe ciphers, and remove the deci-

---

* The reafon of this Rule is evident ; for fince the divifor multiplied by the quotient gives the dividend, therefore the number of decimal places in the dividend is equal to thofe in the divifor and quotient, taken together, by the nature of Multiplication; and confequently the quotient itfelf muft contain as many as the dividend exceeds the divifor.

mal

mal point in the dividend as many places farther to the left as there are ciphers cut off, prefixing ciphers if need be; then proceed as before*.

## EXAMPLES.

1. Divide 45·5 by 2100.

21·00 ) ·455 ( ·0216 &c.
           35
          140
           14

2. Divide 41020 by 32000.
3. Divide    953 by 21600.
4. Divide      61 by 79000.

### CONTRACTION II.

HENCE, if the divisor be 1 with ciphers, as 10, or 100, or 1000, &c : then the quotient will be found by merely moving the decimal point in the dividend so many places farther to the left as the divisor has ciphers; prefixing ciphers if need be.

## EXAMPLES.

So, 217·3 ÷ 100 = 2·173.      And 419 ÷ 10 =
And 5·16 ÷ 100 =         .      And ·21 ÷ 1000 =

### CONTRACTION III.

WHEN there are many figures in the divisor; or when only a certain number of decimals are necessary to be retained in the quotient; then take only as many figures of the divisor as will be equal to the number of figures, both integers and decimals, to be in the quotient, and find how many times they may be contained in the first figures of the dividend, as usual.

---

* This is no more than dividing both divisor and dividend by the same number, either 10, or 100, or 1000, &c, according to the number of ciphers cut off; which, leaving them in the same proportion, does not affect the quotient. And in the same way, the decimal point may be moved the same number of places in both the divisor and dividend, either to the right or left, whether they have ciphers or not.

Let

Let each remainder be a new dividend; and for every such dividend, leave out one figure more on the right-hand side of the divifor ; remembering to carry for the increafe of the figures cut off, as in the 2d contraction in Multiplication.

*Note.* When there are not fo many figures in the divifor as are required to be in the quotient, begin the operation with all the figures, and continue it as ufual till the number of figures in the divifor be equal to thofe remaining to be found in the quotient, after which begin the contraction.

### E X A M P L E S.

1. Divide 2508·92806 by 92·41035, fo as to have only four decimals in the quotient, in which cafe the quotient will contain fix figures.

| *Contracted.* | *Common way.* |
|---|---|
| 92·4103,5)2508·928,06(27·1498 | 92·4103,5)2508·928,06(27·1498 |
| 660721 | 66072106 |
| 13849 | 13848610 |
| 4608 | 46075750 |
| 912 | 91116100 |
| 80 | 79467850 |
| 6 | 5539570 |

2. Divide 4109·2351 by 230·409, fo that the quotient may contain only four decimals.

3. Divide 37·10438 by 5713·96, that the quotient may contain only five decimals.

4. Divide 913·08 by 2137·2, that the quotient may contain only three decimals.

---

## REDUCTION of DECIMALS.

### C A S E I.

*To reduce a Vulgar Fraction to its equivalent Decimal.*

### R U L E.

DIVIDE the numerator by the denominator as in Divifion of Decimals, annexing ciphers to the numerator as far as neceffary ; fo fhall the quotient be the decimal required.

EXAM-

## EXAMPLES.

1. Reduce $\frac{7}{24}$ to a decimal.

$$24 = 4 \times 6. \quad \text{Then } 4\ )\ 7\cdot$$
$$6\ )\ 1\cdot750000$$
$$\cdot291666\ \&c.$$

2. Reduce $\frac{1}{4}$, and $\frac{1}{2}$, and $\frac{3}{4}$ to decimals.

Anf. ·25, and ·5, and ·75.

3. Reduce $\frac{3}{8}$ to a decimal.   Anf. ·375.
4. Reduce $\frac{1}{25}$ to a decimal.   Anf. ·04.
5. Reduce $\frac{3}{192}$ to a decimal.   Anf. ·015625.
6. Reduce $\frac{275}{3842}$ to a decimal.   Anf. ·071577 &c.

## CASE II.

*To find the Value of a Decimal in terms of the Inferior Denominations.*

### RULE.

MULTIPLY the decimal by the number of parts in the next lower denomination; and cut off as many places for a remainder, to the right-hand, as there are places in the given decimal.

Multiply that remainder by the parts in the next lower denomination again, cutting off for another remainder as before.

Proceed in the fame manner through all the parts of the integer; then the feveral denominations feparated on the left-hand, will make up the anfwer.

*Note,* This operation is the fame as Reduction Defcending in whole numbers.

### EXAMPLES.

1. Required to find the value of ·775 pounds fterling.

$$\cdot775$$
$$20$$
$$\overline{\phantom{xx}}$$
$$s\ 15\cdot500$$
$$12 \qquad \text{Anf. 15s 6d.}$$
$$\overline{\phantom{xx}}$$
$$d\ 6\cdot000$$

Ex. 2.

Ex. 2. What is the value of ·625 fhil ?    Anf. 7½d.
3.  What is the value of ·8635l ?    Anf. 17s 3·24d.
4.  What is the value of ·0125lb troy ?    Anf. 3 dwts.
5.  What is the value of ·4694lb troy ?

Anf. 5oz 12dwt 15·744gr.

6.  What is the value of ·625 cwt ?    Anf. 2qr 14lb.
7.  What is the value of ·009943 miles ?

Anf. 17yd 1ft 5·98848inc.

8.  What is the value of ·6875 yd ?    Anf. 2qr 3nls.
9.  What is the value of ·3375 acr ?    Anf. 1rd 14poles.
10. What is the value of ·2083hhd of wine ?

Anf. 13·1229gal.

## CASE III.

*To reduce Integers or Decimals to Equivalent Decimals of Higher Denominations.*

### RULE.

DIVIDE by the number of parts in the next higher denomination; continuing the operation to as many higher denominations as may be neceffary, the fame as in Reduction Afcending of whole numbers.

### EXAMPLES.

1. Reduce 1dwt. to the decimal of a pound troy.

    20 | 1 dwt
    12 | 0·05 oz
        | 0·004166 &c 1b Anf.

2.  Reduce 9d to the decimal of a pound.    Anf. ·0375l.
3.  Reduce 7 drams to the decimal of a pound avoird.

Anf. ·02734375lb.

4.  Reduce ·26d to the decimal of a l.    Anf. ·0010833 &c l.
5.  Reduce 2·15lb to the decimal of a cwt.

Anf. ·019196 + cwt.

6.  Reduce 24 yards to the decimal of a mile.

Anf. ·013636 &c mile.

7.  Reduce ·056 pole to the decimal of an acre.

Anf. ·00035 ac.

8.  Reduce 1·2 pint of wine to the decimal of a hhd.

Anf. ·00238 + hhd.

9.  Reduce 14 minutes to the decimal of a day.

Anf. ·009722 &c da.

10. Reduce ·21 pint to the decimal of a peck.

Anf. ·013125 pec.

11. Reduce 28 ″ 12 ‴ to the decimal of a minute.

NOTE

Note, *When there are several numbers, to be reduced all to the decimal of the highest :*

Set the given numbers directly under each other, for dividends, proceeding orderly from the loweſt denomination to the higheſt.

Oppoſite to each dividend, on the left-hand, ſet ſuch a number for a diviſor as will bring it to the next higher name ; drawing a perpendicular line between all the diviſors and dividends.

Begin at the uppermoſt, and perform all the diviſions ; only obſerving to ſet the quotient of each diviſion, as decimal parts, on the right-hand of the dividend next below it ; ſo ſhall the laſt quotient be the decimal required.

## EXAMPLES.

1. Reduce 15s 9¾d to the decimal of a pound.

$$
\begin{array}{r|l}
4 & 3 \\
12 & 9\ 75 \\
20 & 15\!\cdot\!8125 \\
\end{array}
$$

£ 0·790625 Anſ.

2. Reduce 19l 17s 3¼d to l.     Anſ. 19·86354166 &c l.
3. Reduce 15s 6d to the decimal of a l.     Anſ. ·775l.
4. Reduce 7½d to the decimal of a ſhilling    Anſ. ·625s.
5. Reduce 5oz 12dwts 16gr to lbs.     Anſ. ·46944 &c lb.

## RULE-of-THREE in DECIMALS.

### RULE.

PREPARE the terms by reducing, the vulgar fractions to decimals, any compound numbers either to decimals of the higher denominations, or to integers of the lower, alſo the firſt and third terms to the ſame name : Then multiply and divide as in whole numbers.

*Note*, Any of the convenient Examples in the Rule-of-Three or Rule-of-Five in Integers, or Vulgar Fractions, may be taken as proper examples to the ſame rules in Decimals. —The following Example, which is the firſt in Vulgar Fractions, is wrought out here, to ſhew the method.

If $\frac{3}{8}$ of a yard of velvet coft $\frac{2}{5}$l. what will $\frac{5}{16}$yd. coft ?

yd 　 l 　 yd 　 l 　 s d

$\frac{3}{8} = \cdot 375$ 　 $\cdot 375 : \cdot 4 :: \cdot 3125 : \cdot 333$ &c. or 6 8

$\cdot 4$

$\frac{2}{5} = \cdot 4$ 　 $\cdot 375\ )\ \cdot 12500\ \ (\cdot 333333$ &c.

1250 　 20

125 　 s6·66666 &c.

$\frac{5}{16} = \cdot 3125$ 　 12

Anf. 6s 8d. 　 d7·99999 &c = 8d

---

# DUODECIMALS.

DUODECIMALS, or CROSS MULTIPLICATION, is a rule made ufe of by workmen and artificers, in computing the contents of their works.

Dimenfions are ufually taken in feet, inches, and quarters; any parts fmaller than thefe being neglected as of no confequence. And the fame in mutiplying them together, or cafting up the contents.

## RULE.

SET down the two dimenfions, to be multiplied together, one under the other, fo that feet ftand under feet, inches under inches, &c.

Multiply each term in the multiplicand, beginning at the loweft, by the feet in the multiplier, and fet the refult of each ftraight under its correfponding term, obferving to carry 1 for every 12, from the inches to the feet.

In like manner, multiply all the multiplicand by the inches and parts of the multiplier, and fet the refult of each term one place removed to the right-hand of thofe in the multiplicand; omitting however what is below parts of inches, only carrying to thefe the proper number of units from the loweft denomination.

Or, inftead of multiplying by the inches, take fuch parts of the multiplicand as thefe are of a foot.

Then add the two lines together, after the manner of Compound Addition, carrying 1 to the feet for 12 inches, when thefe come to fo many.

EXAM-

## EXAMPLES.

1. Multiply 4f 7inc          2. Multiply 14f 9inc.
      by 6  4                        by  4  6
      ─────                          ─────
        27  6                          59  0
         1  6⅓                          7  4½
    Anf. ─────                      Anf. ─────
    Anf. 29  0⅓                     Anf. 66  4½

3. Multiply 4 feet 7 inches by 9f 6inc.    Anf.  43f 6½inc.
4. Multiply 12f 5inc by 6f 8inc.           Anft  82  9⅓
5. Multiply 35f 4½inc by 12f 3inc:         Anf. 433  4⅛
6. Multiply 64f 6inc by 8f 9¼inc.          Anf. 565  8⅝

———————————

# INVOLUTION.

INVOLUTION is the raifing of Powers from any given number, as a root.

A Power is a quantity produced by multiplying any given number, called the Root, a certain number of times continually by itfelf. Thus,

$$2 = \ 2 \text{ is the root, or 1ft power of 2.}$$
$$2 \times 2 = \ 4 \text{ is the 2d power, or fquare of 2.}$$
$$2 \times 2 \times 2 = \ 8 \text{ is the 3d power, or cube of 2.}$$
$$2 \times 2 \times 2 \times 2 = 16 \text{ is the 4th power of 2, &c.}$$

And in this manner may be calculated the following Table of the firft nine powers of the firft 9 numbers.

## TABLE of the firſt NINE POWERS of NUMBERS.

| 1ſt | 2d | 3d | 4th | 5th | 6th | 7th | 8th | 9th |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 |
| 3 | 9 | 27 | 81 | 243 | 729 | 2187 | 6561 | 19683 |
| 4 | 16 | 64 | 256 | 1024 | 4096 | 16384 | 65536 | 262144 |
| 5 | 25 | 125 | 625 | 3125 | 15625 | 78125 | 390625 | 1953125 |
| 6 | 36 | 216 | 1296 | 7776 | 46656 | 279936 | 1679616 | 10077696 |
| 7 | 49 | 343 | 2401 | 16807 | 117649 | 823543 | 5764801 | 40353607 |
| 8 | 64 | 512 | 4096 | 32768 | 262144 | 2097152 | 16777216 | 134217728 |
| 9 | 81 | 729 | 6561 | 59049 | 531441 | 4782969 | 43046721 | 387420489 |

The Index or Exponent of a Power, is the number denoting the height or degree of that power; and it is 1 more than the number of multiplications uſed in producing the ſame. So 1 is the index or exponent of the 1ſt power or root, 2 of the 2d power or ſquare, 3 of the 3d power or cube, 4 of the 4th power, and ſo on.

Powers, that are to be raiſed, are uſually denoted by placing the index above the root or firſt power.

$$So \quad 2^2 = 4 \text{ is the 2d power of 2.}$$
$$2^3 = 8 \text{ is the 3d power of 2.}$$
$$2^4 = 16 \text{ is the 4th power of 2.}$$
$$540^4 \qquad \text{is the 4th power of 540, &c.}$$

When two or more powers are multiplied together, their product will be that power whoſe index is the ſum of the exponents of the factors or powers multiplied. Or the multiplication of the powers, anſwers to the addition of the indices. Thus, in the following powers of 2,

| 1ft | 2d | 3d | 4th | 5th | 6th | 7th | 8th | 9th | 10th |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 | 1024 |
| or $2^1$ | $2^2$ | $2^3$ | $2^4$ | $2^5$ | $2^6$ | $2^7$ | $2^8$ | $2^9$ | $2^{10}$ |

Here, $4 \times 4 = 16$, and $2 + 2 = 4$ its index;
and $8 \times 16 = 128$, and $3 + 4 = 7$ its index;
alfo $16 \times 64 = 1024$, and $4 + 6 = 10$ its index;

### OTHER EXAMPLES.

1. What is the 2d power of 45 ?  Anf. 2025.
2. What is the fquare of 4·16 ?  Anf. 17·3056.
3. What is the 3d power of 3·5 ?  Anf. 42·875.
4. What is the 5th power of ·029 ? Anf. ·00000020511149.
5. What is the fquare of $\frac{2}{3}$ ?  Anf. $\frac{4}{9}$.
6. What is the 3d power of $\frac{5}{9}$ ?  Anf. $\frac{125}{729}$.
7. What is the 4th power of $\frac{3}{4}$ ?  Anf. $\frac{81}{256}$.

# EVOLUTION.

Evolution, or the reverfe of Involution, is the extract-
ing or finding the roots of any given powers.

The root of any number, or power, is fuch a number, as
being multiplied into itfelf a certain number of times, will
produce that power. Thus, 2 is the fquare-root or 2d root
of 4, becaufe $2^2 = 2 \times 2 = 4$; and 3 is the cube-root or 3d
root of 27, becaufe $3^3 = 3 \times 3 \times 3 = 27$.

Any power of a given number or root may be found ex-
actly, namely, by multiplying the number continually into
itfelf. But there are many numbers of which a propofed root
can never be exactly found. Yet, by means of decimals we
may approximate or approach towards the root, to any de-
gree of exactnefs.

Thofe roots which only approximate, are called Surd
roots ; but thofe which can be found quite exact, are called
Rational roots. Thus, the fquare root of 3 is a furd root ;
but the fquare root of 4 is a rational root, being equal to 2 ;
alfo the cube root of 8 is rational, being equal to 2 ; but the
cube root of 9 is furd or irrational.

Roots are fometimes denoted by writing the character $\sqrt{}$
before the power, with the index of the root againft it.

Thus,

Thus, the 3d root of 20 is expreffed by $\sqrt[3]{20}$; and the fquare root or 2d root of it is $\sqrt{20}$, the index 2 being always omit‑ ted, when the fquare root is defigned.

When the power is expreffed by feveral numbers, with the fign $+$ or $-$ between them, a line is drawn from the top of the fign over all the parts of it : thus the third root of $45 - 12$ is $\sqrt[3]{45-12}$, or thus $\sqrt[3]{}(45-12)$ inclofing the numbers in parenthefes.

But all roots are now often defigned like powers with fractional indices : thus, the fquare root of 8 is $8^{\frac{1}{2}}$, the cube root of 25 is $25^{\frac{1}{3}}$, and the 4th root of $45 - 18$ is $\overline{45-18}|^{\frac{1}{4}}$, or $(45-18)^{\frac{1}{4}}$.

## TO EXTRACT THE SQUARE ROOT.

### RULE.*

DIVIDE the given number into periods of two figures each, by fetting a point over the place of units, another over the place of hundreds, and fo on, over every fecond figure, both to the left-hand in integers, and to the right in decimals.

Find

---

* The reafon for feparating the figures of the dividend into pe‑ riods or portions of two places each, is, that the fquare of any fin‑ gle figure never confifts of more than two places; the fquare of a number of two figures, of not more than four places, and fo on. So that there will be as many figures in the root as the given num‑ ber contains periods fo divided or parted off.

And the reafon of the feveral fteps in the operation, appears from the algebraic form of the fquare of any number of terms, whether two or three or more. Thus,

$\overline{a+b}|^2 = a^2 + 2ab + b^2 = a^2 + \overline{2a+b} \cdot b$, the fquare of two terms; where it appears that $a$ is the firft term of the root, and $b$ the fecond term; alfo $a$ the firft divifor, and the new divifor is $2a+b$, or double the firft term increafed by the fecond. And hence the manner of extraction is thus:

1ft divifor $a$ ) $a^2 + 2ab + b^2$ ( $a + b$ the root.
$\qquad a^2$
$\qquad \overline{\phantom{aaaa}}$
2d divifor $2a + b$ | $2ab + b^2$
$\qquad b$ | $2ab + b^2$
$\qquad \overline{\phantom{aaaa}}$

Find the greateſt ſquare in the firſt period on the left-hand, and ſet its root on the right-hand of the given number, after the manner of a quotient figure in Diviſion.

Subtract the ſquare thus found from the ſaid period, and to the remainder annex the two figures of the next following period, for a dividend.

Double the root above mentioned for a diviſor; and find how often it is contained in the ſaid dividend, excluſive of its right-hand figure; and ſet that quotient figure both in the quotient and diviſor.

Multiply the whole augmented diviſor by this laſt quotient figure, and ſubtract the product from the ſaid dividend, bringing down to it the next period of the given number, for a new dividend.

Repeat the ſame proceſs over again, viz. find another new diviſor, by doubling all the figures now found in the root; from which, and the laſt dividend, find the next figure of the root as before; and ſo on through all the periods, to the laſt.

*Note,* The beſt way of doubling the root, to form the new diviſors, is by adding the laſt figure always to the laſt diviſor, as appears in the following examples.—Alſo, after the figures belonging to the given number are all exhauſted, the operation may be continued into decimals at pleaſure, by adding any number of periods of ciphers, two in each period.

---

Again, for a root of three parts *a*, *b*, *c*, thus :

$$\overline{a + b + c}\,|^2 = a^2 + 2ab + b^2 + 2ac + 2bc + c^2 =$$
$$a^2 + \overline{2a + b}.\,b + \overline{2a + 2b + c}.\,c,\text{ the}$$

ſquare of three terms; where *a* is the firſt term of the root, *b* the ſecond, and *c* the third term; alſo *a* the firſt diviſor, $2a + b$ the ſecond, and $2a + 2b + c$ the third, each conſiſting of the double of the root increaſed by the next term of the ſame. And the mode of extraction is thus :

1ſt diviſor $a\,)\,a^2 + 2ab + b^2 + 2ac + 2bc + c^2\,(a + b + c$ the root.
$$a^2$$

---

2d diviſor $2a + b$ | $2ab + b^2$
$\qquad\qquad\quad b$ | $2ab + b^2$

---

3d diviſor $2a + 2b + c$ | $2ac + 2bc + c^2$
$\qquad\qquad\qquad\quad c$ | $2ac + 2bc + c^2.$

---

EXAMPLES.

1. To find the fquare root of 29506624.

```
        . . . .
    29506624 ( 5432 the root.
    25
104 | 450
  4 | 416
1083 | 3466
   3 | 3249
10862 | 21724
    2 | 21724
```

NOTE, *When the root is to be extracted to many places of figures, the work may be confiderably fhortened, thus:*

Having proceeded in the extraction after the common method, till there be found half the required number of figures in the root, or one figure more; then, for the reft, divide the laft remainder by its correfponding divifor after the manner of the third contraction in Divifion of Decimals ; thus,

2. To find the root of 2 to nine places of figures.

```
        2 ( 1·4142
        I
    24 | 100
     4 | 96
   281 | 400
     1 | 281
  2824 | 11900
     4 | 11296
 28282 | 60400
     2 | 56564
 28284 ) 3836 ( 1356
          1008
           160
            19
             2
```

Anf. 1·41421356 the root required.

3. What

3. What is the fquare root of 2025 ?      Anf. 45.
4. What is the fquare root of 17·3056 ?      Anf. 4·16.
5. What is the fquare root of ·000729 ?      Anf. ·027.
6. What is the fquare root of 3 ?      Anf. 1·732050.
7. What is the fquare root of 5 ?      Anf. 2·236068.
8. What is the fquare root of 6 ?      Anf. 2·449489.
9. What is the fquare root of 7 ?      Anf. 2·645751.
10. What is the fquare root of 10 ?      Anf. 3·162277.
11. What is the fquare root of 11 ?      Anf. 3·316624.
12. What is the fquare root of 12 ?      Anf. 3·464101.

### RULES FOR THE SQUARE ROOTS OF VULGAR FRACTIONS AND MIXED NUMBERS.

First prepare all vulgar fractions, by reducing them to their leaft terms, both for this and all other roots. Then

1. Take the root of the numerator and of the denominator for the refpective terms of the root required. And this is the beft way if the denominator be a complete power : but if it be not, then

2. Multiply the numerator and denominator together ; take the root of the product : this root being made the numerator to the denominator of the given fraction, or made the denominator to the numerator of it, will form the fractional root required.

$$\text{That is, } \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{ab}}{b} = \frac{a}{\sqrt{ab}}.$$

And this rule will ferve whether the root be finite or infinite.

3. Or reduce the vulgar fraction to a decimal, and extract its root.

4. Mixed numbers may be either reduced to improper fractions, and extracted by the firft or fecond rule ; or the vulgar fraction may be reduced to a decimal, then joined to the integer, and the root of the whole extracted.

### EXAMPLES.

1. What is the root of $2\frac{5}{36}$ ?      Anf. $\frac{5}{6}$.
2. What is the root of $\frac{27}{147}$ ?      Anf. $\frac{3}{7}$.
3. What is the root of $\frac{9}{12}$ ?      Anf. 0·866025.
4. What is the root of $\frac{5}{12}$ ?      Anf. 0·645497.
5. What is the root of $17\frac{3}{8}$ ?      Anf. 4·168333.

By means of the fquare root alfo may readily be found the 4th root, or the 8th root, or the 16th root, &c ; that is, the root of any power whofe index is fome power of the number 2 ;

number 2 ; namely, by extracting fo often the fquare root as is denoted by that power of 2 ; that is, two extractions for the fourth root, three for the 8th root, and fo on.

So, to find the 4th root of the number 21035·8, extract the fquare root two times as follows :

$$\begin{array}{lll}
21035\cdot8000 & (\;145\cdot037237\;(12\cdot0431407\;\text{the 4th root.} \\
\;\;\;\underline{1} & \;\;\;\underline{1}
\end{array}$$

| 24 | 110 | | 22 | 45 |
|----|-----|--|----|----|
| 4 | 96 | | 2 | 44 |
| 285 | 1435 | | 2404 | 10372 |
| 5 | 1425 | | 4 | 9616 |
| 29003 | 108000 | 24083 | | 75639 |
| 6 | 87009 | 6 | | 72249 |
| | 20991 (7237 | | 3388 (1407 |
| | 687 | | 980 |
| | 107 | | 17 |

Ex. 2. What is the 4th root of 97·41 ?

## TO EXTRACT THE CUBE ROOT.

### I. *By the Common Rule\*.*

1. HAVING divided the given number into periods of three figures each, (by fetting a point over the place of units, and alfo over every third figure, from thence, to the left hand in whole numbers, and to the right in decimals) find the neareft lefs cube to the firft period ; fet its root in the quotient, and fubtract the faid cube from the firft period ; to the remainder bring down the fecond period, and call this the refolvend.

---

* The reafon for pointing the given number into periods of three figures each, is becaufe the cube of one figure never amounts to more than three places. And, for a fimilar reafon, a given number is pointed into periods of four figures for the 4th root, of five figures for the 5th root, and fo on.

And the reafon for the other parts of the rule depends on the algebraic formation of a cube : for, if the root confift of the two parts $a + b$, then its cube is as follows : $\overline{a + b}|^3 = a^3 + 3a^2b + 3ab^2 + b^3$ ; where $a$ is the root of the firft part $a^3$ ; the refolvend is $3a^2b + 3ab^2 + b^3$, which is alfo the fame as the three parts of the fubtrahend ; alfo the divifor is $3a^2 + 3a$, by which dividing the firft two terms of the refolvend $3a^2b + 3ab^2$, gives $b$ for the fecond part of the root ; and fo on.

2. To

2. To three times the fquare of the root, juft found, add three times the root itfelf, fetting this one place more to the right than the former, and call this fum the divifor. Then divide the refolvend, wanting the laft figure, by the divifor, for the next figure of the root, which annex to the former; calling this laft figure *e*, and the part of the root before found let be called *a*.

3. Add all together thefe three products, namely, thrice *a* fquare multiplied by *e*, thrice *a* multiplied by *e* fquare, and *e* cube, fetting each of them one place more to the right than the former, and call the fum the fubtrahend; which muft not exceed the refolvend; but if it does, then make the laft figure *e* lefs, and repeat the operation for finding the fubtrahend, till it be lefs than the refolvend.

4. From the refolvend take the fubtrahend, and to the remainder join the next period of the given number for a new refolvend; to which form a new divifor from the whole root now found; and from thence another figure of the root, as directed in Article 2, and fo on.

## EXAMPLE.

To extract the cube root of 48228·544.

$$3 \times 3^2 = 27 \quad | \quad 48228\cdot544 \ (36\cdot4 \text{ root.}$$
$$3 \times 3 = 09 \quad | \quad 27$$

Divifor 279 | 21228 refolvend.

$$\left.\begin{array}{l} 3 \times 3^2 \times 6 = 162 \\ 3 \times 3 \times 6^2 = 324 \\ 6^3 = 216 \end{array}\right\} \text{add}$$

$$3 \times 36^2 = 3888 \quad | \quad 19656 \text{ fubtrahend.}$$
$$3 \times 36 = 108$$

38988 | 1572544 refolvend.

$$\left.\begin{array}{l} 3 \times 36^2 \times 4 = 15552 \\ 3 \times 36 \times 4^2 = 1728 \\ 4^3 = 64 \end{array}\right\} \text{add}$$

1572544 fubtrahend.

0000000 remainder.

Ex. 2. Extract the cube root of 571482·19.
Ex. 3. Extract the cube root of 1628·1582.
Ex. 4. Extract the cube root of 1332.

II. To

## II. *To extract the Cube Root by a short Way\*.*

1. By trials, or by the table of roots at pa. 90, take the neareſt rational cube to the given number, whether it be greater or leſs; and call it the aſſumed cube.

2. Then ſay, by the Rule-of-Three, As the ſum of the given number and double the aſſumed cube, is to the ſum of the aſſumed cube and double the given number, ſo is the root of the aſſumed cube, to the root required, nearly. Or, As the firſt ſum is to the difference of the given and aſſumed cube, ſo is the aſſumed root, to the difference of the roots nearly.

3. Again, by uſing, in like manner, the cube of the root laſt found as a new aſſumed cube, another root will be obtained ſtill nearer. And ſo on as far as we pleaſe; uſing always the cube of the laſt found root, for the aſſumed cube.

### EXAMPLE.

To find the Cube root of 21035·8.

Here we ſoon find that the root lies between 20 and 30, and then between 27 and 28. Taking therefore 27, its cube is 19683, which is the aſſumed cube. Then

$$
\begin{array}{ll}
19683 & 21035\text{·}8 \\
\phantom{1968}2 & \phantom{2103}2 \\
\hline
39366 & 42071\text{·}6 \\
21035\text{·}8 & 19683 \\
\hline
\text{As } 60401\text{·}8 \;:\; 61754\text{·}6 \;::\; 27 \;:\; 27\text{·}6047 \\
\phantom{As } 60401\text{·}8 \;:\; \phantom{61754}27 \\
\phantom{As } 60401\text{·}8 \;:\; \overline{4322822} \\
\phantom{As } 60401\text{·}8 \;:\; \phantom{4}1235092
\end{array}
$$

$$
60401\text{·}8 \;\big)\; 1667374\text{·}2 \;\big(\; 27\text{·}6047 \text{ the root nearly}
$$
$$
\begin{array}{r}
459338 \\
36525 \\
284 \\
42
\end{array}
$$

Again,

---

\* The method uſually given for extracting the cube root, is ſo exceedingly tedious, and difficult to be remembered, that various other approximating rules have been invented, by Newton, Raphſon, Halley, De Lagny, Simpſon, Emerſon, and ſeveral other

mathe-

Again, for a fecond operation, the cube of this root is 21035˙318645155823, and the procefs by the latter method will be thus:

　　　21035˙318645 &c
　　　　　　　2
─────────────
　　42070˙637290　　21035˙8
　　21035˙8　　　　　21035˙318645 &c.
　─────────────　　　─────────────

As 63106˙43729　:　diff. ˙481355 :: 27˙6047 :
　　　　　　　　　　　　the dif.　　　˙0002 10834
　　　　　　　　　　　　　　　　　─────────────
　　　　　　confeq. the root req. is 27˙604910834

Ex. 2. To extract the cube root of ˙67.
Ex. 3. To extract the cube root of ˙01.

### TO EXTRACT ANY ROOT WHATEVER*.

LET P be the given power or number, $n$ the index of the power, A the affumed power, $r$ its root, R the required root of P.

Then, as the fum of $n + 1$ times A and $n - 1$ times P, is to the fum of $n + 1$ times P and $n - 1$ times A, fo is the affumed root $r$, to the required root R.

Or, as half the faid fum of $n + 1$ times A and $n - 1$ times P, is to the difference between the given and affumed powers, fo is the affumed root $r$, to the difference between the true and affumed roots: which difference, added or fubtracted, as the cafe requires, gives the true root nearly.

That is, $\overline{n + 1} . A + \overline{n - 1} . P : \overline{n + 1} . P + \overline{n - 1} . A :: r : R.$

Or, $\overline{n + 1} . \frac{1}{2}A + \overline{n - 1} . \frac{1}{2}P : P \backsim A :: r : R \backsim r.$

────────────────────────────

mathematicians; but no one that I have yet feen, is fo fimple in its form, or feems fo well adapted for general ufe, as that above given. This rule, as far as I have learned, firft came from Mr. James Dodfon, and is the fame in effect as Dr. Halley's rational formula, but fomewhat more commodioufly exprelfed; and the firft inveftigation of it was given in my Tracts, pa. 49. The algebraic form of it is this:

　　　　As P + 2A : A + 2P :: r : R. Or,
　　　　As P + 2A : P ∽ A :: r : R ∽ r;

where P is the given number, A the affumed neareft cube, $r$ the cube root of A, and R the root of P fought.

* This is a very general approximating rule, of which that for the cube root is a particular cafe, and is the beft adapted for practice, and for memory, of any that I have yet feen. It was firft difcovered in this form by myfelf, and the inveftigation and ufe of it were given at large in my Tracts, pa. 45, &c.

　　　　　　　　　　　　　　　　　　　　　　　And

And the operation may be repeated as often as we pleafe, by ufing always the laft found root for the affumed root, and its $n$th power for the affumed power A.

### EXAMPLE.

*To extract the 5th root of 21035·8.*

Here it appears that the 5th root is between 7·3 and 7·4. Taking 7·3, its 5th power is 20730·71593. Hence we have, P $=$ 21035·8, $n=5$, $r=$ 7·3 and A $=$ 20730·71593; then $\overline{n+1}.\frac{1}{2}$ A $+$ $\overline{n-1}.\frac{1}{2}$ P : P $\wp$ A $::$ $r$ : R $\wp$ $r$, that is,

$3\times 20730\cdot71593 + 2\times 21035\cdot8 : 305\cdot084 :: 7\cdot3 :$

$$\quad\quad\quad\quad\quad 3 \quad\quad\quad\quad\quad 2 \quad\quad\quad 7\cdot3$$

| 62192·14779 | 42071·6 | 915252 |
| 42071·6 | | 2135588 |

104263·74779  )  2227·1132( ·0213605=R $\wp$ $r$
7·3 $=$ $r$, add.

7·321360 $=$ R, true
to the laft figure.

### OTHER EXAMPLES.

1. What is the 3d root of 2 ?     Anf. 1.259921,
2. What is the 4th root of 2 ?     Anf. 1·189207.
3. What is the 4th root of 97·41 ?     Anf. 3·1415999.
4. What is the 5th root of 2 ?     Anf. 1·148699.
5. What is the 6th root of 21035·8 ?     Anf. 5·254037.
6. What is the 6th root of 2 ?     Anf. 1·122462.
7. What is the 7th root of 21035·8 ?     Anf. 4·145392.
8. What is the 7th root of 2 ?     Anf. 1·104089.
9. What is the 8th root of 21035·8 ?     Anf. 3·470323.
10. What is the 8th root of 2 ?     Anf. 1·090508.
11. What is the 9th root of 21035·8 ?     Anf. 3·022239.
12. What is the 9th root of 2 ?     Anf. 1.080059.

A TABLE

A TABLE of SQUARES and CUBES, also SQUARE ROOTS and CUBE ROOTS.

| Number. | Square. | Cube. | Square Root. | Cube Root. |
|---|---|---|---|---|
| 1 | 1 | 1 | 1·0000000 | 1·000000 |
| 2 | 4 | 8 | 1·4142136 | 1·259921 |
| 3 | 9 | 27 | 1·7320508 | 1·442250 |
| 4 | 16 | 64 | 2·0000000 | 1·587401 |
| 5 | 25 | 125 | 2·2360680 | 1·709976 |
| 6 | 36 | 216 | 2·4494897 | 1·817121 |
| 7 | 49 | 343 | 2·6457513 | 1·912933 |
| 8 | 64 | 512 | 2·8284271 | 2·000000 |
| 9 | 81 | 729 | 3·0000000 | 2·080084 |
| 10 | 100 | 1000 | 3·1622777 | 2·154435 |
| 11 | 121 | 1331 | 3·3166248 | 2·223980 |
| 12 | 144 | 1728 | 3·4641016 | 2·289428 |
| 13 | 169 | 2197 | 3·6055513 | 2·351335 |
| 14 | 196 | 2744 | 3·7416574 | 2·410142 |
| 15 | 225 | 3375 | 3·8729833 | 2·466212 |
| 16 | 256 | 4096 | 4·0000000 | 2·519842 |
| 17 | 289 | 4913 | 4·1231056 | 2·571282 |
| 18 | 324 | 5832 | 4·2426407 | 2·620741 |
| 19 | 361 | 6859 | 4·3588989 | 2·668402 |
| 20 | 400 | 8000 | 4·4721360 | 2·714418 |
| 21 | 441 | 9261 | 4·5825757 | 2·758923 |
| 22 | 484 | 10648 | 4·6904158 | 2·802039 |
| 23 | 529 | 12167 | 4·7958315 | 2·843867 |
| 24 | 576 | 13824 | 4·8989795 | 2·884499 |
| 25 | 625 | 15625 | 5·0000000 | 2·924018 |
| 26 | 676 | 17576 | 5·0990195 | 2·962496 |
| 27 | 729 | 19683 | 5·1961524 | 3·000000 |
| 28 | 784 | 21952 | 5·2915026 | 3·036589 |
| 29 | 841 | 24389 | 5·3851648 | 3·072317 |
| 30 | 900 | 27000 | 5·4772256 | 3·107232 |
| 31 | 961 | 29791 | 5·5677644 | 3·141381 |
| 32 | 1024 | 32768 | 5·6568542 | 3·174802 |
| 33 | 1089 | 35937 | 5·7445626 | 3·207534 |
| 34 | 1156 | 39304 | 5·8309519 | 3·239612 |
| 35 | 1225 | 42875 | 5·9160798 | 3·271066 |
| 36 | 1296 | 46656 | 6·0000000 | 3·301927 |
| 37 | 1369 | 50653 | 6·0827625 | 3·332222 |
| 38 | 1444 | 54872 | 6·1644140 | 3·361975 |
| 39 | 1521 | 59319 | 6·2449980 | 3·391211 |
| 40 | 1600 | 64000 | 6·3245553 | 3·419952 |

| Number. | Square. | Cube. | Square Root. | Cube Root. |
|---|---|---|---|---|
| 41 | 1681 | 68921 | 6·4031242 | 3·448217 |
| 42 | 1764 | 74088 | 6·4807407 | 3·476027 |
| 43 | 1849 | 79507 | 6·5574385 | 3·503398 |
| 44 | 1936 | 85184 | 6·6332496 | 3·530348 |
| 45 | 2025 | 91125 | 6·7082039 | 3·556893 |
| 46 | 2116 | 97336 | 6·7823300 | 3·583048 |
| 47 | 2209 | 103823 | 6·8556546 | 3·608826 |
| 48 | 2304 | 110592 | 6·9282032 | 3·634241 |
| 49 | 2401 | 117649 | 7·0000000 | 3·659306 |
| 50 | 2500 | 125000 | 7·0710678 | 3·684031 |
| 51 | 2601 | 132651 | 7·1414284 | 3·708430 |
| 52 | 2704 | 140608 | 7·2111026 | 3·732511 |
| 53 | 2809 | 148877 | 7·2801099 | 3·756286 |
| 54 | 2916 | 157464 | 7·3484692 | 3·779763 |
| 55 | 3025 | 166375 | 7·4161985 | 3·802953 |
| 56 | 3136 | 175616 | 7·4893148 | 3·825862 |
| 57 | 3249 | 185193 | 7·5498344 | 3·848561 |
| 58 | 3364 | 195112 | 7·6157731 | 3·870877 |
| 59 | 3481 | 205379 | 7·6811457 | 3·892996 |
| 60 | 3600 | 216000 | 7·7459667 | 3·914867 |
| 61 | 3721 | 226981 | 7·8102497 | 3·936497 |
| 62 | 3844 | 238328 | 7·8740079 | 3·957892 |
| 63 | 3969 | 250047 | 7·9372539 | 3·979057 |
| 64 | 4096 | 262144 | 8·0000000 | 4·000000 |
| 65 | 4225 | 274625 | 8·0622577 | 4·020726 |
| 66 | 4356 | 287496 | 8·1240384 | 4·041240 |
| 67 | 4489 | 300763 | 8·1853528 | 4·061548 |
| 68 | 4624 | 314432 | 8·2462113 | 4·081656 |
| 69 | 4761 | 328509 | 8·3066239 | 4·101566 |
| 70 | 4900 | 343000 | 8·3666003 | 4·121285 |
| 71 | 5041 | 357911 | 8·4261498 | 4·140818 |
| 72 | 5184 | 373248 | 8·4852814 | 4·160168 |
| 73 | 5329 | 389017 | 8·5440037 | 4·179339 |
| 74 | 5476 | 405224 | 8·6023253 | 4·198336 |
| 75 | 5625 | 421875 | 8·6602540 | 4·217163 |
| 76 | 5776 | 438976 | 8·7177979 | 4·235824 |
| 77 | 5929 | 456533 | 8·7749644 | 4·254321 |
| 78 | 6084 | 474552 | 8·8317609 | 4·272659 |
| 79 | 6241 | 493039 | 8·8881944 | 4·290841 |
| 80 | 6400 | 512000 | 8·9442719 | 4·308870 |
| 81 | 6561 | 531441 | 9·0000000 | 4·326749 |
| 82 | 6724 | 551368 | 9·0553851 | 4·344481 |
| 83 | 6889 | 571787 | 9·1104336 | 4·362071 |
| 84 | 7056 | 592704 | 9·1651514 | 4·379519 |
| 85 | 7225 | 614125 | 9·2195445 | 4·396830 |

| Number. | Square. | Cube. | Square Root. | Cube Root. |
|---|---|---|---|---|
| 86 | 7396 | 636056 | 9·2736185 | 4·414005 |
| 87 | 7569 | 658503 | 9·3273791 | 4·431047 |
| 88 | 7744 | 681472 | 9·3808315 | 4 447960 |
| 89 | 7921 | 704969 | 9·4339811 | 4·464745 |
| 90 | 8100 | 729000 | 9·4868330 | 4·481405 |
| 91 | 8281 | 753571 | 9·5393920 | 4 497942 |
| 92 | 8464 | 778688 | 9·5916630 | 4·514357 |
| 93 | 8649 | 804357 | 9·6436508 | 4·530655 |
| 94 | 8836 | 830584 | 9·6953597 | 4·546836 |
| 95 | 9025 | 857375 | 9·7467943 | 4·562903 |
| 96 | 9216 | 884736 | 9·7979590 | 4·578857 |
| 97 | 9409 | 912673 | 9·8488578 | 4·594701 |
| 98 | 9604 | 941192 | 9·8994949 | 4·610436 |
| 99 | 9801 | 970299 | 9·9498744 | 4·626065 |
| 100 | 10000 | 1000000 | 10·0000000 | 4·641589 |
| 101 | 10201 | 1030301 | 10·0498756 | 4·657010 |
| 102 | 10404 | 1061208 | 10·0995049 | 4·672330 |
| 103 | 10609 | 1092727 | 10·1488916 | 4 687548 |
| 104 | 10816 | 1124864 | 10·1980390 | 4·702669 |
| 105 | 11025 | 1157625 | 10·2469508 | 4·717694 |
| 106 | 11236 | 1191016 | 10·2956301 | 4·732624 |
| 107 | 11449 | 1225043 | 10·3440804 | 4·747459 |
| 108 | 11664 | 1259712 | 10·3923048 | 4·762203 |
| 109 | 11881 | 1295029 | 10·4403065 | 4·776856 |
| 110 | 12100 | 1331000 | 10·4880885 | 4·791420 |
| 111 | 12321 | 1367631 | 10·5356538 | 4·805896 |
| 112 | 12544 | 1404928 | 10·5830052 | 4·820284 |
| 113 | 12769 | 1442897 | 10·6301458 | 4·834588 |
| 114 | 12996 | 1481544 | 10·6770783 | 4·848808 |
| 115 | 13225 | 1520875 | 10·7238053 | 4·862944 |
| 116 | 13456 | 1560896 | 10·7703296 | 4·876999 |
| 117 | 13689 | 1601613 | 10·8166538 | 4·890973 |
| 118 | 13924 | 1643032 | 10·8627805 | 4·904868 |
| 119 | 14161 | 1685159 | 10·9087121 | 4·918685 |
| 120 | 14400 | 1728000 | 10·9544512 | 4·932424 |
| 121 | 14641 | 1771561 | 11·0000000 | 4·946088 |
| 122 | 14884 | 1815848 | 11·0453610 | 4·959675 |
| 123 | 15129 | 1860867 | 11·0905365 | 4·973190 |
| 124 | 15376 | 1906624 | 11·1355287 | 4·986631 |
| 125 | 15625 | 1953125 | 11·1803399 | 5·000000 |
| 126 | 15876 | 2000376 | 11·2249722 | 5·013298 |
| 127 | 16129 | 2048383 | 11·2694277 | 5·026526 |
| 128 | 16384 | 2097152 | 11·3137085 | 5·039684 |
| 129 | 16641 | 2146689 | 11·3578167 | 5·052774 |
| 130 | 16900 | 2197000 | 11·4017543 | 5·065797 |

| Number. | Square. | Cube. | Square Root. | Cube Root. |
|---|---|---|---|---|
| 131 | 17161 | 2248091 | 11·4455231 | 5·078753 |
| 132 | 17424 | 2299968 | 11·4891253 | 5·091643 |
| 133 | 17689 | 2352637 | 11·5325626 | 5·104469 |
| 134 | 17956 | 2406104 | 11·5758369 | 5·117230 |
| 135 | 18225 | 2460375 | 11·6189500 | 5·129928 |
| 136 | 18496 | 2515456 | 11·6619038 | 5·142563 |
| 137 | 18769 | 2571353 | 11·7046999 | 5·155137 |
| 138 | 19044 | 2628072 | 11·7473444 | 5·167649 |
| 139 | 19321 | 2685619 | 11·7898261 | 5·180101 |
| 140 | 19600 | 2744000 | 11·8321596 | 5·192494 |
| 141 | 19881 | 2803221 | 11·8743421 | 5·204828 |
| 142 | 20164 | 2863288 | 11·9163753 | 5·217103 |
| 143 | 20449 | 2924207 | 11·9582607 | 5·229321 |
| 144 | 20736 | 2985984 | 12·0000000 | 5·241482 |
| 145 | 21025 | 3048625 | 12·0415946 | 5·253588 |
| 146 | 21316 | 3112136 | 12·0830460 | 5·265637 |
| 147 | 21609 | 3176523 | 12·1243557 | 5·277632 |
| 148 | 21904 | 3241792 | 12·1655251 | 5·289572 |
| 149 | 22201 | 3307949 | 12·2065556 | 5·301459 |
| 150 | 22500 | 3375000 | 12·2474487 | 5·313293 |
| 151 | 22801 | 3442951 | 12·2882057 | 5·325074 |
| 152 | 23104 | 3511808 | 12·3288280 | 5·336803 |
| 153 | 23409 | 3581577 | 12·3693169 | 5·348481 |
| 154 | 23716 | 3652264 | 12·4096736 | 5·360108 |
| 155 | 24025 | 3723875 | 12·4498996 | 5·371685 |
| 156 | 24336 | 3796416 | 12·4899960 | 5·383213 |
| 157 | 24649 | 3869893 | 12·5299641 | 5·394690 |
| 158 | 24964 | 3944312 | 12·5698051 | 5·406120 |
| 159 | 25281 | 4019679 | 12·6095202 | 5·417501 |
| 160 | 25600 | 4096000 | 12·6491106 | 5·428835 |
| 161 | 25921 | 4173281 | 12·6885775 | 5·440122 |
| 162 | 26244 | 4251528 | 12·7279221 | 5·451362 |
| 163 | 26569 | 4330747 | 12·7671453 | 5·462556 |
| 164 | 26896 | 4410944 | 12·8062485 | 5·473703 |
| 165 | 27225 | 4492125 | 12·8452326 | 5·484806 |
| 166 | 27556 | 4574296 | 12·8840987 | 5·495865 |
| 167 | 27889 | 4657463 | 12·9228480 | 5·506879 |
| 168 | 28224 | 4741632 | 12·9614814 | 5·517848 |
| 169 | 28561 | 4826809 | 13·0000000 | 5·528775 |
| 170 | 28900 | 4913000 | 13·0384048 | 5·539658 |
| 171 | 29241 | 5000211 | 13·0766968 | 5·550499 |
| 172 | 29584 | 5088448 | 13·1148770 | 5·561298 |
| 173 | 29929 | 5177717 | 13·1529464 | 5·572054 |
| 174 | 30276 | 5268024 | 13·1909060 | 5·582770 |
| 175 | 30625 | 5359375 | 13·2287566 | 5·593445 |

| Num-ber. | Square. | Cube. | Square Root. | Cube Root. |
|---|---|---|---|---|
| 176 | 30976 | 5451776 | 13.2664992 | 5.604079 |
| 177 | 31329 | 5545233 | 13.3041347 | 5.614673 |
| 178 | 31684 | 5639752 | 13.3416641 | 5.625226 |
| 179 | 32041 | 5735339 | 13.3790882 | 5.635741 |
| 180 | 32400 | 5832000 | 13.4164079 | 5.646216 |
| 181 | 32761 | 5929741 | 13.4536240 | 5.656652 |
| 182 | 33124 | 6028568 | 13.4907376 | 5.667051 |
| 183 | 33489 | 6128487 | 13.5277493 | 5.677411 |
| 184 | 33856 | 6229504 | 13.5646600 | 5.687734 |
| 185 | 34225 | 6331625 | 13.6014705 | 5.698019 |
| 186 | 34596 | 6434856 | 13.6381817 | 5.708267 |
| 187 | 34969 | 6539203 | 13.6747943 | 5.718479 |
| 188 | 35344 | 6644672 | 13.7113092 | 5.728654 |
| 189 | 35721 | 6751269 | 13.7477271 | 5.738794 |
| 190 | 36100 | 6859000 | 13.7840488 | 5.748897 |
| 191 | 36481 | 6967871 | 13.8202750 | 5.758965 |
| 192 | 36864 | 7077888 | 13.8564065 | 5.768998 |
| 193 | 37429 | 7189057 | 13.8924440 | 5.778996 |
| 194 | 37636 | 7301384 | 13.9283883 | 5.788960 |
| 195 | 38025 | 7414875 | 13.9642400 | 5.798890 |
| 196 | 38416 | 7529536 | 14.0000000 | 5.808786 |
| 197 | 38809 | 7645373 | 14.0356688 | 5.818648 |
| 198 | 39204 | 7762392 | 14.0712473 | 5.828476 |
| 199 | 39601 | 7880599 | 14.1067360 | 5.838272 |
| 200 | 40000 | 8000000 | 14.1421356 | 5.848035 |
| 201 | 40401 | 8120601 | 14.1774469 | 5.857765 |
| 202 | 40804 | 8242408 | 14.2126704 | 5.867464 |
| 203 | 41209 | 8365427 | 14.2478068 | 5.877130 |
| 204 | 41616 | 8489664 | 14.2828569 | 5.886765 |
| 205 | 42025 | 8615125 | 14.3178211 | 5.896368 |
| 206 | 42436 | 8741816 | 14.3527001 | 5.905941 |
| 207 | 42849 | 8869743 | 14.3874946 | 5.915481 |
| 208 | 43264 | 8998912 | 14.4222051 | 5.924991 |
| 209 | 43681 | 9123329 | 14.4568323 | 5.934473 |
| 210 | 44100 | 9261000 | 14.4913767 | 5.943911 |
| 211 | 44521 | 9393931 | 14.5258390 | 5.953341 |
| 212 | 44944 | 9528128 | 14.5602198 | 5.962731 |
| 213 | 45369 | 9663597 | 14.5945195 | 5.972091 |
| 214 | 45796 | 9800344 | 14.6287388 | 5.981426 |
| 215 | 46225 | 9938375 | 14.6628783 | 5.990727 |
| 216 | 46656 | 10077696 | 14.6969385 | 6.000000 |
| 217 | 47089 | 10218313 | 14.7309199 | 6.009244 |
| 218 | 47524 | 10360282 | 14.7648231 | 6.018463 |
| 219 | 47961 | 10503459 | 14.7986486 | 9.027650 |
| 220 | 48400 | 10648000 | 14.8323970 | 9.036811 |

| Number. | Square. | Cube. | Square Root. | Cube Root. |
|---|---|---|---|---|
| 221 | 48841 | 10793861 | 14·8660687 | 6·045943 |
| 222 | 49284 | 10941048 | 14·8996644 | 6·055048 |
| 223 | 49729 | 11089567 | 14·9331845 | 6·064126 |
| 224 | 50176 | 11239424 | 14·9666295 | 6·073177 |
| 225 | 50625 | 11390625 | 15·0000000 | 6·082201 |
| 226 | 51076 | 11543176 | 15·0332964 | 6·091199 |
| 227 | 51529 | 11697083 | 15·0665192 | 6·100170 |
| 228 | 51984 | 11852352 | 15·0996689 | 6·109115 |
| 229 | 52441 | 12008989 | 15·1327460 | 6·118032 |
| 230 | 52900 | 12167000 | 15·1657509 | 6·126925 |
| 231 | 53361 | 12326391 | 15·1986842 | 6·135792 |
| 232 | 53824 | 12487168 | 15·2315462 | 6·144634 |
| 233 | 54289 | 12649337 | 15·2643375 | 6·153449 |
| 234 | 54756 | 12812904 | 15·2970585 | 6·162239 |
| 235 | 55225 | 12977875 | 15·3297097 | 6·171005 |
| 236 | 55696 | 13144256 | 15·3622915 | 6·179747 |
| 237 | 56169 | 13312053 | 15·3948043 | 6·188463 |
| 238 | 56644 | 13481272 | 15·4272486 | 6·197154 |
| 239 | 57121 | 13651919 | 15·4596248 | 6·205821 |
| 240 | 57600 | 13824000 | 15·4919334 | 6·214464 |
| 241 | 58081 | 13997521 | 15·5241747 | 6·223083 |
| 242 | 58564 | 14172488 | 15·5563492 | 6·231678 |
| 243 | 59049 | 14348907 | 15·5884573 | 6·240251 |
| 244 | 59536 | 14526784 | 15·6204994 | 6·248800 |
| 245 | 60025 | 14706125 | 15·6524758 | 6·257324 |
| 246 | 60516 | 14886936 | 15·6843871 | 6·265826 |
| 247 | 61009 | 15069223 | 15·7162336 | 6·274304 |
| 248 | 61504 | 15252992 | 15·7480157 | 6·282760 |
| 249 | 62001 | 15438249 | 15·7797338 | 6·291194 |
| 250 | 62500 | 15625000 | 15·8113883 | 6·299604 |
| 251 | 63001 | 15813251 | 15·8429795 | 6·307992 |
| 252 | 63504 | 16003008 | 15·8745079 | 6·316359 |
| 253 | 64009 | 16194277 | 15·9059737 | 6·324704 |
| 254 | 64516 | 16387064 | 15·9373775 | 6·333025 |
| 255 | 65025 | 16581375 | 15·9687194 | 6·341325 |
| 256 | 65536 | 16777216 | 16·0000000 | 6·349602 |
| 257 | 66049 | 16974593 | 16·0312195 | 6·357859 |
| 258 | 66564 | 17173512 | 16·0623784 | 6·366095 |
| 259 | 67081 | 17373979 | 16·0934769 | 6·374310 |
| 260 | 67600 | 17576000 | 16·1245155 | 6·382504 |
| 261 | 68121 | 17779581 | 16·1554944 | 6·390676 |
| 262 | 68644 | 17984728 | 16·1864141 | 6·398827 |
| 263 | 69169 | 18191447 | 16·2172747 | 6·406958 |
| 264 | 69696 | 18399744 | 16·2480768 | 6·415068 |
| 265 | 70225 | 18609625 | 16·2788206 | 6·423157 |
| 266 | 70756 | 18821096 | 16·3095064 | 6·431226 |

| Number. | Square. | Cube. | Square Root. | Cube Root. |
|---|---|---|---|---|
| 267 | 71289 | 19034163 | 16·3401346 | 6·439275 |
| 268 | 71824 | 19248832 | 16·3707055 | 6·447305 |
| 269 | 72361 | 19465109 | 16·4012195 | 6·455314 |
| 270 | 72900 | 19683000 | 16·4316767 | 6·463304 |
| 271 | 73441 | 19902511 | 16·4620776 | 6·471274 |
| 272 | 73984 | 20123648 | 16·4924225 | 6·479224 |
| 273 | 74529 | 20346417 | 16·5227116 | 6·487153 |
| 274 | 75076 | 20570824 | 16·5529454 | 6·495064 |
| 275 | 75625 | 20796875 | 16·5831240 | 6·502956 |
| 276 | 76176 | 21024576 | 16·6132477 | 6·510829 |
| 277 | 76729 | 21253933 | 16·6433170 | 6·518684 |
| 278 | 77284 | 21484952 | 16·6733320 | 6·526519 |
| 279 | 77841 | 21717639 | 16·7032931 | 6·534335 |
| 280 | 78400 | 21952000 | 16·7332005 | 6·542132 |
| 281 | 78961 | 22188041 | 16·7630546 | 6·549911 |
| 282 | 79524 | 22425768 | 16·7928556 | 6·557672 |
| 283 | 80089 | 22665187 | 16·8226038 | 6·565415 |
| 284 | 80656 | 22906304 | 16·8522995 | 6·573139 |
| 285 | 81225 | 23149125 | 16·8819430 | 6·580844 |
| 286 | 81796 | 23393656 | 16·9115345 | 6·588531 |
| 287 | 82369 | 23639903 | 16·9410743 | 6·596202 |
| 288 | 82944 | 23887872 | 16·9705627 | 6·603854 |
| 289 | 83521 | 24137569 | 17·0000000 | 6·611488 |
| 290 | 84100 | 24389000 | 17·0293864 | 6·619106 |
| 291 | 84681 | 24642171 | 17·0587221 | 6·626705 |
| 292 | 85264 | 24897088 | 17·0880075 | 6·634287 |
| 293 | 85849 | 25153757 | 17·1172428 | 6·641851 |
| 294 | 86436 | 25412184 | 17·1464282 | 6·649399 |
| 295 | 87025 | 25672375 | 17·1755640 | 6·656930 |
| 296 | 87616 | 25934336 | 17·2046505 | 6·664443 |
| 297 | 88209 | 26198073 | 17·2336879 | 6·671940 |
| 298 | 88804 | 26463592 | 17·2626765 | 6·679419 |
| 299 | 89401 | 26730899 | 17·2916165 | 6·686882 |
| 300 | 90000 | 27000000 | 17·3205081 | 6·694328 |
| 301 | 90601 | 27270901 | 17·3493516 | 6·701758 |
| 302 | 91204 | 27543608 | 17·3781472 | 6·709172 |
| 303 | 91809 | 27818127 | 17·4068952 | 6·716569 |
| 304 | 92416 | 28094464 | 17·4355958 | 6·723950 |
| 305 | 93025 | 28372625 | 17·4642492 | 6·731316 |
| 306 | 93636 | 28652616 | 17·4928557 | 6·738665 |
| 307 | 94249 | 28934443 | 17·5214155 | 6·745997 |
| 308 | 94864 | 29218112 | 17·5499288 | 6·753313 |
| 309 | 95481 | 29503629 | 17·5783958 | 6·760614 |
| 310 | 96100 | 29791000 | 17·6068169 | 6·767899 |
| 311 | 96721 | 30080231 | 17·6351921 | 6·775168 |
| 312 | 97344 | 30371328 | 17·6635217 | 6·782422 |

| Number. | Square. | Cube. | Square Root. | Cube Root. |
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| 313 | 97969 | 30664297 | 17·6918060 | 6·789661 |
| 314 | 98596 | 30959144 | 17·7200451 | 6·796884 |
| 315 | 99225 | 31255875 | 17·7482393 | 6·804091 |
| 316 | 99856 | 31554496 | 17·7763888 | 6·811284 |
| 317 | 100489 | 31855013 | 17·8044938 | 6·818461 |
| 318 | 101124 | 32157432 | 17·8325545 | 6·825624 |
| 319 | 101761 | 32461759 | 17·8605711 | 6·832771 |
| 320 | 102400 | 32768000 | 17·8885438 | 6·839903 |
| 321 | 103041 | 33076161 | 17·9164729 | 6·847021 |
| 322 | 103684 | 33386248 | 17·9443584 | 6·854124 |
| 3 3 | 104329 | 33698267 | 17·9722008 | 6·861211 |
| 3 4 | 104976 | 34012224 | 18·0000000 | 6·868284 |
| 325 | 105625 | 34328125 | 18·0277564 | 6·875343 |
| 326 | 106276 | 34645976 | 18·0554701 | 6·882388 |
| 327 | 106929 | 34965783 | 18·0831413 | 6·889419 |
| 328 | 107584 | 35287552 | 18·1107703 | 6·896435 |
| 329 | 108241 | 35611289 | 18·1383571 | 6·903436 |
| 330 | 108900 | 35937000 | 18·1659021 | 6·910423 |
| 331 | 109561 | 36264691 | 18·1934054 | 6·917396 |
| 332 | 110224 | 36594368 | 18·2208672 | 6·924355 |
| 333 | 110889 | 36926037 | 18·2482876 | 6·931300 |
| 334 | 111556 | 37259704 | 18·2756669 | 6·938232 |
| 335 | 112225 | 37595375 | 18·3030052 | 6·945149 |
| 336 | 112896 | 37933056 | 18·3303028 | 6·952053 |
| 337 | 113569 | 38272753 | 18·3575598 | 6·958943 |
| 338 | 114244 | 38614472 | 18·3847763 | 6·965819 |
| 339 | 114921 | 38958219 | 18·4119526 | 6·972682 |
| 340 | 115600 | 39304000 | 18·4390889 | 6·979532 |
| 341 | 116281 | 39651821 | 18·4661853 | 6·986369 |
| 342 | 116964 | 40001688 | 18·4932420 | 6·993191 |
| 343 | 117649 | 40353607 | 18·5202592 | 7·000000 |
| 344 | 118336 | 40707584 | 18·5472370 | 7·006796 |
| 345 | 119025 | 41063625 | 18·5741756 | 7·013579 |
| 346 | 119716 | 41421736 | 18·6010752 | 7·020349 |
| 347 | 120409 | 41781923 | 18·6279360 | 7·027106 |
| 348 | 121104 | 42144192 | 18·6547581 | 7·033850 |
| 349 | 121801 | 42508549 | 18·6815417 | 7·040581 |
| 350 | 122500 | 42875000 | 18·7082869 | 7·047298 |
| 351 | 123201 | 43243551 | 18·7349940 | 7·054003 |
| 352 | 123904 | 43614208 | 18·7616630 | 7·060696 |
| 353 | 124609 | 43986977 | 18·7882942 | 7·067376 |
| 354 | 125316 | 44361864 | 18·8148877 | 7·074043 |
| 355 | 126025 | 44738875 | 18·8414437 | 7·080698 |
| 356 | 126736 | 45118016 | 18·8679623 | 7·087341 |
| 357 | 127449 | 45499293 | 18·8944436 | 7·093970 |
| 358 | 128164 | 45882712 | 18·9208879 | 7·100588 |

| Num-ber | Square. | Cube. | Square Root. | Cube Root. |
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| 359 | 128881 | 46268279 | 18·9472953 | 7·107193 |
| 360 | 129600 | 46656000 | 18·9736660 | 7·113786 |
| 361 | 130321 | 47045881 | 19·0000000 | 7·120367 |
| 362 | 131044 | 47437928 | 19·0262976 | 7·126935 |
| 363 | 131769 | 47832147 | 19·0525589 | 7·133492 |
| 364 | 132496 | 48228544 | 19·0787840 | 4·140037 |
| 365 | 133225 | 48627125 | 19·1049732 | 7·146569 |
| 366 | 133956 | 49027896 | 19·1311265 | 7·153090 |
| 367 | 134689 | 49430863 | 19·1572441 | 7·159599 |
| 368 | 135424 | 49836032 | 19·1833261 | 7·166095 |
| 369 | 136161 | 50243409 | 19·2093727 | 7·172580 |
| 370 | 136900 | 50653000 | 19·2353841 | 7·179054 |
| 371 | 137641 | 51064811 | 19·2613603 | 7·185516 |
| 372 | 138384 | 51478848 | 19·2873015 | 7·191966 |
| 373 | 139129 | 51895117 | 19·3132079 | 7·198405 |
| 374 | 139876 | 52313624 | 19·3390796 | 7·204832 |
| 375 | 140625 | 52734375 | 19·3649167 | 7·211247 |
| 376 | 141376 | 53157376 | 19·3907194 | 7·217652 |
| 377 | 142129 | 53582633 | 19·4164878 | 7·224045 |
| 378 | 142884 | 54010152 | 19·4422221 | 7·230427 |
| 379 | 143641 | 54439939 | 19·4679223 | 7·236797 |
| 380 | 144400 | 54872000 | 19·4935887 | 7·243156 |
| 381 | 145161 | 55306341 | 19·5192213 | 7·249504 |
| 382 | 145924 | 55742968 | 19·5448203 | 7·255841 |
| 383 | 146689 | 56181887 | 19·5703858 | 7·262167 |
| 384 | 147456 | 56623104 | 19·5959179 | 7·268482 |
| 385 | 148225 | 57066625 | 19·6214169 | 7·274786 |
| 386 | 148996 | 57512456 | 19·6468827 | 7·281079 |
| 387 | 149769 | 57960603 | 19·6723156 | 7·287362 |
| 388 | 150544 | 58411072 | 19·6977156 | 7·293633 |
| 389 | 151321 | 58863869 | 19·7230829 | 7·299893 |
| 390 | 152100 | 59319000 | 19·7484177 | 7·306143 |
| 391 | 152881 | 59776471 | 19·7737199 | 7·312383 |
| 392 | 153664 | 60236288 | 19·7989899 | 7·318611 |
| 393 | 154449 | 60698457 | 19·8242276 | 7·324829 |
| 394 | 155236 | 61162984 | 19·8494332 | 7·331037 |
| 395 | 156025 | 61629875 | 19·8746069 | 7·337234 |
| 396 | 156816 | 62099136 | 19·8997487 | 7·343420 |
| 397 | 157609 | 62570773 | 19·9248588 | 7·349596 |
| 398 | 158404 | 63044792 | 19·9499373 | 7·355762 |
| 399 | 159201 | 63521199 | 19·9749844 | 7·361917 |
| 400 | 160000 | 64000000 | 20·0000000 | 7·368063 |
| 401 | 160801 | 64481201 | 20·0249844 | 7·374198 |
| 402 | 161604 | 64964808 | 20·0499377 | 7·380322 |
| 403 | 162409 | 65450827 | 20·0748599 | 7·386437 |
| 404 | 163216 | 65939264 | 20·0997512 | 7·392542 |

| Number. | Square. | Cube. | Square Root. | Cube Root. |
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| 405 | 164025 | 66430125 | 20·1246118 | 7·398636 |
| 406 | 164836 | 66923416 | 20·1494417 | 7·404720 |
| 407 | 165649 | 67419143 | 20·1742410 | 7·410794 |
| 408 | 166464 | 67911312 | 20·1990099 | 7·416859 |
| 409 | 167281 | 68417929 | 20·2237484 | 7·422914 |
| 410 | 168100 | 68921000 | 20·2484567 | 7·428958 |
| 411 | 168921 | 69426531 | 20·2731349 | 7·434993 |
| 412 | 169744 | 69934528 | 20·2977831 | 7·441018 |
| 413 | 170569 | 70444997 | 20·3224014 | 7·447033 |
| 414 | 171396 | 70957944 | 20·3469899 | 7·453039 |
| 415 | 172225 | 71473375 | 20·3715488 | 7·459036 |
| 416 | 173056 | 71991296 | 20·3960781 | 7·465022 |
| 417 | 173889 | 72511713 | 20·4205779 | 7·470999 |
| 418 | 174724 | 73034632 | 20·4450483 | 7·476966 |
| 419 | 175561 | 73560059 | 20·4694895 | 7·482924 |
| 420 | 176400 | 74088000 | 20·4939015 | 7·488872 |
| 421 | 177241 | 74618461 | 20·5182845 | 7·494810 |
| 422 | 178084 | 75151448 | 20·5426386 | 7·500740 |
| 423 | 178929 | 75686967 | 20·5669638 | 7·506660 |
| 424 | 179776 | 76225024 | 20·5912603 | 7·512571 |
| 425 | 180625 | 76765625 | 20·6155281 | 7·518473 |
| 426 | 181476 | 77308776 | 20·6397674 | 7·524365 |
| 427 | 182329 | 77854483 | 20·6639783 | 7·530248 |
| 428 | 183184 | 78402752 | 20·6881609 | 7·536121 |
| 429 | 184041 | 78953589 | 20·7123152 | 7·541986 |
| 430 | 184900 | 79507000 | 20·7364414 | 7·547841 |
| 431 | 185761 | 80062991 | 20·7605395 | 7·553688 |
| 432 | 186624 | 80621568 | 20·7846097 | 7·559525 |
| 433 | 187489 | 81182737 | 20·8086520 | 7·565353 |
| 434 | 188356 | 81746504 | 20·8326667 | 7·571173 |
| 435 | 189225 | 82312875 | 20·8566536 | 7·576984 |
| 436 | 190096 | 82881856 | 20·8806130 | 7·582786 |
| 437 | 190969 | 83453453 | 20·9045450 | 7·588579 |
| 438 | 191844 | 84027672 | 20·9284495 | 7·594363 |
| 439 | 192721 | 84604519 | 20·9523268 | 7·600138 |
| 440 | 193600 | 85184000 | 20·9761770 | 7·605905 |
| 441 | 194481 | 85766121 | 21·0000000 | 7·611662 |
| 442 | 195364 | 86350888 | 21·0237960 | 7·617411 |
| 443 | 196249 | 86938307 | 21·0475652 | 7·623151 |
| 444 | 197136 | 87528384 | 21·0713075 | 7·628883 |
| 445 | 198025 | 88121125 | 21·0950231 | 7·634606 |
| 446 | 198916 | 88716536 | 21·1187121 | 7·640321 |
| 447 | 199809 | 89314623 | 21·1423745 | 7·646027 |
| 448 | 200704 | 89915392 | 21·1660105 | 7·651725 |
| 449 | 201601 | 90518849 | 21·1896201 | 7·657414 |
| 450 | 202500 | 91125000 | 21·2132034 | 7·663094 |

| Number. | Square. | Cube. | Square Root. | Cube Root. |
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| 451 | 203401 | 91733851 | 21·2367606 | 7·668766 |
| 452 | 204304 | 92345408 | 21·2602916 | 7·674430 |
| 453 | 205209 | 92959677 | 21·2837967 | 7·680085 |
| 454 | 206116 | 93576664 | 21·3072758 | 7·685732 |
| 455 | 207025 | 94196375 | 21·3307290 | 7·691371 |
| 456 | 207936 | 94818816 | 21·3541565 | 7·697002 |
| 457 | 208849 | 95443993 | 21·3775583 | 7·702624 |
| 458 | 209764 | 96071912 | 21·4009346 | 7·708238 |
| 459 | 210681 | 96702579 | 21·4242853 | 7·713844 |
| 460 | 211600 | 97336000 | 21·4476106 | 7·719442 |
| 461 | 212521 | 97972181 | 21·4709106 | 7·725032 |
| 462 | 213444 | 98611128 | 21·4941853 | 7·730614 |
| 463 | 214369 | 99252847 | 21·5174348 | 7·736187 |
| 464 | 215296 | 99897344 | 21·5406592 | 7·741753 |
| 465 | 216225 | 100544625 | 21·5638587 | 7·747310 |
| 466 | 217156 | 101194696 | 21·5870331 | 7·752860 |
| 467 | 218089 | 101847563 | 21·6101828 | 7·758402 |
| 468 | 219024 | 102503232 | 21·6333077 | 7·763936 |
| 469 | 219961 | 103161709 | 21·6564078 | 7·769462 |
| 470 | 220900 | 103823000 | 21·6794834 | 7·774980 |
| 471 | 221841 | 104487111 | 21·7025344 | 7·780490 |
| 472 | 222784 | 105154048 | 21·7255610 | 7·785992 |
| 473 | 223729 | 105823817 | 21·7485632 | 7·791487 |
| 474 | 224676 | 106496424 | 21·7715411 | 7·796974 |
| 475 | 225625 | 107171875 | 21·7944947 | 7·802453 |
| 476 | 226576 | 107850176 | 21·8174242 | 7·807925 |
| 477 | 227529 | 108531333 | 21·8403297 | 7·813389 |
| 478 | 228484 | 109215352 | 21·8632111 | 7·818845 |
| 479 | 229441 | 109902239 | 21·8860686 | 7·824294 |
| 480 | 230400 | 110592000 | 21·9089023 | 7·829735 |
| 481 | 231361 | 111284641 | 21·9317122 | 7·835168 |
| 482 | 232324 | 111980168 | 21·9544984 | 7·840594 |
| 483 | 233289 | 112678587 | 21·9772610 | 7·846013 |
| 484 | 234256 | 113379904 | 22·0000000 | 7·851424 |
| 485 | 235225 | 114084125 | 22·0227155 | 7·856828 |
| 486 | 236196 | 114791256 | 22·0454077 | 7·862224 |
| 487 | 237169 | 115501303 | 22·0680765 | 7·867613 |
| 488 | 238144 | 116214272 | 22·0907220 | 7·872994 |
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| 491 | 241081 | 118370771 | 22·1585198 | 7·889094 |
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| 493 | 243049 | 119823157 | 22·2036033 | 7·899791 |
| 494 | 244036 | 120553784 | 22·2261108 | 7·905129 |
| 495 | 245025 | 121287375 | 22·2485955 | 7·910460 |
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| Number. | Square. | Cube. | Square Root. | Cube Root. |
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| 497 | 247009 | 122763473 | 22·2934968 | 7·921100 |
| 498 | 248004 | 123505992 | 22·3159136 | 7·926408 |
| 499 | 249001 | 124251499 | 22·3383079 | 7·931710 |
| 500 | 250000 | 125000000 | 22·3606798 | 7·937005 |
| 501 | 251001 | 125751501 | 22·3830293 | 7·942293 |
| 502 | 252004 | 126506008 | 22·4053565 | 7·947573 |
| 503 | 253009 | 127263527 | 22·4276615 | 7·952847 |
| 504 | 254016 | 128024064 | 22·4499443 | 7·958114 |
| 505 | 255025 | 128787625 | 22·4722051 | 7·963374 |
| 506 | 256036 | 129554216 | 22·4944438 | 7·968627 |
| 507 | 257049 | 130323843 | 22·5166605 | 7·973873 |
| 508 | 258064 | 131096512 | 22·5388553 | 7·979112 |
| 509 | 259081 | 131872229 | 22·5610283 | 7·984344 |
| 510 | 260100 | 132651000 | 22·5831796 | 7·989569 |
| 511 | 261121 | 133432831 | 22·6053091 | 7·994788 |
| 512 | 262144 | 134217728 | 22·6274170 | 8·000000 |
| 513 | 263169 | 135005697 | 22·6495033 | 8·005205 |
| 514 | 264196 | 135796744 | 22·6715681 | 8·010403 |
| 515 | 265225 | 136590875 | 22·6936114 | 8·015595 |
| 516 | 266256 | 137388096 | 22·7156334 | 8·020779 |
| 517 | 267289 | 138188413 | 22·7376340 | 8·025957 |
| 518 | 268324 | 138991832 | 22·7596134 | 8·031129 |
| 519 | 269361 | 139798359 | 22·7815715 | 8·036293 |
| 520 | 270400 | 140608000 | 22·8035085 | 8·041451 |
| 521 | 271441 | 141420761 | 22·8254244 | 8·046603 |
| 522 | 272484 | 142236648 | 22·8473193 | 8·051748 |
| 523 | 273529 | 143055667 | 22·8691933 | 8·056886 |
| 524 | 274576 | 143877824 | 22·8910463 | 8·062018 |
| 525 | 275625 | 144703125 | 22·9128785 | 8·067143 |
| 526 | 276676 | 145531576 | 22·9346899 | 8·072262 |
| 527 | 277729 | 146363183 | 22·9564806 | 8·077374 |
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| 530 | 280900 | 148877000 | 23·0217289 | 8·092672 |
| 531 | 281961 | 149721291 | 23·0434372 | 8·097758 |
| 532 | 283024 | 150568768 | 23·0651252 | 8·102838 |
| 533 | 284089 | 151419437 | 23·0867928 | 8·107912 |
| 534 | 285156 | 152273304 | 23·1084400 | 8·112980 |
| 535 | 286225 | 153130375 | 23·1300670 | 8·118041 |
| 536 | 287296 | 153990656 | 23·1516738 | 8·123096 |
| 537 | 288369 | 154854153 | 23·1732605 | 8·128144 |
| 538 | 289444 | 155720872 | 23·1948270 | 8·133186 |
| 539 | 290521 | 156590819 | 23·2163735 | 8·138223 |
| 540 | 291600 | 157464000 | 23·2379001 | 8·143253 |
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| Number. | Square. | Cube. | Square Root. | Cube Root. |
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| 543 | 294849 | 160103007 | 23·3023604 | 8·158304 |
| 544 | 295936 | 160989184 | 23·3238076 | 8·163309 |
| 545 | 297025 | 161878625 | 23·3452351 | 8·168308 |
| 546 | 298116 | 162771336 | 23·3666429 | 8·173302 |
| 547 | 299209 | 163667323 | 23·3880311 | 8 178289 |
| 548 | 300304 | 164566592 | 23·4093998 | 8 183269 |
| 549 | 301401 | 165469149 | 23·4307490 | 8·188244 |
| 550 | 302500 | 166375000 | 23·4520788 | 8·193212 |
| 551 | 303601 | 167284151 | 23·4733892 | 8·198175 |
| 552 | 304704 | 168196608 | 23·4946802 | 8·203131 |
| 553 | 305809 | 169112377 | 23·5159520 | 8·208082 |
| 554 | 306916 | 170031464 | 23·5372046 | 8·213027 |
| 555 | 308025 | 170953875 | 23·5584380 | 8·217965 |
| 556 | 309136 | 171879616 | 23·5796522 | 8·222898 |
| 557 | 310249 | 172808693 | 23·6008474 | 8·227825 |
| 558 | 311364 | 173741112 | 23·6220236 | 8·232746 |
| 559 | 312481 | 174676879 | 23·6431808 | 8·237661 |
| 560 | 313600 | 175616000 | 23·6643191 | 8·242570 |
| 561 | 314721 | 176558481 | 23·6854386 | 8·247474 |
| 562 | 315844 | 177504328 | 23·7065392 | 8·252371 |
| 563 | 316969 | 178453547 | 23·7276210 | 8·257263 |
| 564 | 318096 | 179406144 | 23·7486842 | 8·262149 |
| 565 | 319225 | 180362125 | 23·7697286 | 8·267029 |
| 566 | 320356 | 181321496 | 23·7907545 | 8·271903 |
| 567 | 321489 | 182284263 | 23·8117618 | 8·276772 |
| 568 | 322624 | 183250432 | 23·8327506 | 8·281635 |
| 569 | 323761 | 184220009 | 23·8537209 | 8·286493 |
| 570 | 324900 | 185193000 | 23·8746728 | 8·291344 |
| 571 | 326041 | 186169411 | 23·8956063 | 8·296190 |
| 572 | 327184 | 187149248 | 23·9165215 | 8·301030 |
| 573 | 328329 | 188132517 | 23·9374184 | 8·305865 |
| 574 | 329476 | 189119224 | 23·9582971 | 8·310694 |
| 575 | 330625 | 190109375 | 23·9791576 | 8·315517 |
| 576 | 331776 | 191102976 | 24·0000000 | 8·320335 |
| 577 | 332929 | 192100033 | 24·0208243 | 8·325147 |
| 578 | 334084 | 193100552 | 24·0416306 | 8·329954 |
| 579 | 335241 | 194104539 | 24·0624188 | 8·334755 |
| 580 | 336400 | 195112000 | 24·0831892 | 8·339551 |
| 581 | 337561 | 196122941 | 24·1039416 | 8·344341 |
| 582 | 338724 | 197137368 | 24·1246762 | 8·349125 |
| 583 | 339889 | 198155287 | 24·1453929 | 8·353904 |
| 584 | 341056 | 199176704 | 24·1660919 | 8·358678 |
| 585 | 342225 | 200201625 | 24·1867732 | 8·363446 |
| 586 | 343396 | 201230056 | 24·2074369 | 8·368209 |
| 587 | 344569 | 202262003 | 24·2280829 | 8·372966 |
| 588 | 345744 | 203297472 | 24·2487113 | 8·377718 |

| Number. | Square. | Cube. | Square Root. | Cube Root. |
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| 589 | 346921 | 204336469 | 24·2693222 | 8·382465 |
| 590 | 348100 | 205379000 | 24·2899156 | 8·387206 |
| 591 | 349281 | 206425071 | 24·3104916 | 8·391942 |
| 592 | 350464 | 207474688 | 24·3310501 | 8·396673 |
| 593 | 351649 | 208527857 | 24·3515913 | 8·401398 |
| 594 | 352836 | 209584584 | 24·3721152 | 8·406118 |
| 595 | 354025 | 210644875 | 24·3926218 | 8·410832 |
| 596 | 355216 | 211708736 | 24·4131112 | 8·415541 |
| 597 | 356409 | 212776173 | 24·4335834 | 8·420245 |
| 598 | 357604 | 213847192 | 24·4540385 | 8·424944 |
| 599 | 358801 | 214921799 | 24·4744765 | 8·429638 |
| 600 | 360000 | 216000000 | 24·4948974 | 8·434327 |
| 601 | 361201 | 217081801 | 24·5153013 | 8·439009 |
| 602 | 362404 | 218167208 | 24·5356883 | 8·443687 |
| 603 | 363609 | 219256227 | 24·5560583 | 8·448360 |
| 604 | 364816 | 220348864 | 24·5764115 | 8·453027 |
| 605 | 366025 | 221445125 | 24·5967478 | 8·457689 |
| 606 | 367236 | 222545016 | 24·6170673 | 8·462347 |
| 607 | 368449 | 223648543 | 24·6373700 | 8·466999 |
| 608 | 369664 | 224755712 | 24·6576560 | 8·471647 |
| 609 | 370881 | 225866529 | 24·6779254 | 8·476289 |
| 610 | 372100 | 226981000 | 24·6981781 | 8·480926 |
| 611 | 373321 | 228099131 | 24·7184142 | 8·485557 |
| 612 | 374544 | 229220928 | 24·7386338 | 8·490184 |
| 613 | 375769 | 230346397 | 24·7588368 | 8·494806 |
| 614 | 376996 | 231475544 | 24·7790234 | 8·499423 |
| 615 | 378225 | 232608375 | 24·7991935 | 8·504034 |
| 616 | 379456 | 233744896 | 24·8193473 | 8·508641 |
| 617 | 380689 | 234885113 | 24·8394847 | 8·513243 |
| 618 | 381924 | 236029032 | 24·8596058 | 8·517840 |
| 619 | 383161 | 237176659 | 24·8797106 | 8·522432 |
| 620 | 384400 | 238328000 | 24·8997992 | 8·527018 |
| 621 | 385641 | 239483061 | 24·9198716 | 8·531600 |
| 622 | 386884 | 240641848 | 24·9399278 | 8·536177 |
| 623 | 388129 | 241804367 | 24·9599679 | 8·540749 |
| 624 | 389376 | 242970624 | 24·9799920 | 8·545317 |
| 625 | 390625 | 244140625 | 25·0000000 | 8·549879 |
| 626 | 391876 | 245314376 | 25·0199920 | 8·554437 |
| 627 | 393129 | 246491883 | 25·0399681 | 8·558990 |
| 628 | 394384 | 247673152 | 25·0599282 | 8·563537 |
| 629 | 395641 | 248858189 | 25·0798724 | 8·568080 |
| 630 | 396900 | 250047000 | 25·0998008 | 8·572618 |
| 631 | 398161 | 251239591 | 25·1197134 | 8·577152 |
| 632 | 399424 | 252435968 | 25·1396102 | 8·581680 |
| 633 | 400689 | 253636137 | 25·1594913 | 8·586204 |
| 634 | 401956 | 254840104 | 25·1793566 | 8·590723 |

| Number. | Square. | Cube. | Square Root. | Cube Root. |
|---|---|---|---|---|
| 635 | 403225 | 256047875 | 25·1992063 | 8·595238 |
| 636 | 404496 | 257259456 | 25·2190404 | 8·599747 |
| 637 | 405769 | 258474853 | 25·2388589 | 8·604252 |
| 638 | 407044 | 259694072 | 25·2586619 | 8·608752 |
| 639 | 408321 | 260917119 | 25·2784493 | 8·613248 |
| 640 | 409600 | 262144000 | 25·2982213 | 8·617738 |
| 641 | 410881 | 263374721 | 25·3179778 | 8·622224 |
| 642 | 412164 | 264609288 | 25·3377189 | 8·626706 |
| 643 | 413449 | 265847707 | 25·3574447 | 8·631183 |
| 644 | 414736 | 267089984 | 25·3771551 | 8·635655 |
| 645 | 416025 | 268336125 | 25·3968502 | 8·640122 |
| 646 | 417316 | 269586136 | 25·4165301 | 8·644585 |
| 647 | 418609 | 270840023 | 25·4361947 | 8·649043 |
| 648 | 419904 | 272097792 | 25·4558441 | 8·653497 |
| 649 | 421201 | 273359449 | 25·4754784 | 8·657946 |
| 650 | 422500 | 274625000 | 25·4950976 | 8·662391 |
| 651 | 423801 | 275894451 | 25·5147016 | 8·666831 |
| 652 | 425104 | 277167808 | 25·5342907 | 8·671266 |
| 653 | 426409 | 278445077 | 25·5538647 | 8·675697 |
| 654 | 427716 | 279726264 | 25·5734237 | 8·680123 |
| 655 | 429025 | 281011375 | 25·5929678 | 8·684545 |
| 656 | 430336 | 282300416 | 25·6124969 | 8·688963 |
| 657 | 431649 | 283593393 | 25·6320112 | 8·693376 |
| 658 | 432964 | 284890312 | 25·6515107 | 8·697784 |
| 659 | 434281 | 286191179 | 25·6709953 | 8·702188 |
| 660 | 435600 | 287496000 | 25·6904652 | 8·706587 |
| 661 | 436921 | 288804781 | 25·7099203 | 8·710982 |
| 662 | 438244 | 290117528 | 25·7293607 | 8·715373 |
| 663 | 439569 | 291434247 | 25·7487864 | 8·719759 |
| 664 | 440896 | 292754944 | 25·7681975 | 8·724141 |
| 665 | 442225 | 294079625 | 25·7875939 | 8·728518 |
| 666 | 443556 | 295408296 | 25·8069758 | 8·732891 |
| 667 | 444889 | 296740963 | 25·8263431 | 8·737260 |
| 668 | 446224 | 298077632 | 25·8456960 | 8·741624 |
| 669 | 447561 | 299418309 | 25·8650343 | 8·745984 |
| 670 | 448900 | 300763000 | 25·8843582 | 8·750340 |
| 671 | 450241 | 302111711 | 25·9036677 | 8·754691 |
| 672 | 451584 | 303464448 | 25·9229628 | 8·759038 |
| 673 | 452929 | 304821217 | 25·9422435 | 8·763380 |
| 674 | 454276 | 306182024 | 25·9615100 | 8·767719 |
| 675 | 455625 | 307546875 | 25·9807621 | 8·772053 |
| 676 | 456976 | 308915776 | 26·0000000 | 8·776382 |
| 677 | 458329 | 310288723 | 26·0192237 | 8·780708 |
| 678 | 459684 | 311665752 | 26·0384331 | 8·785029 |
| 679 | 461041 | 313046839 | 26·0576284 | 8·789346 |
| 680 | 462400 | 314432000 | 26·0768096 | 8·793659 |

| Number. | Square. | Cube. | Square Root. | Cube Root. |
|---|---|---|---|---|
| 681 | 463761 | 315821241 | 26·0959767 | 8·797967 |
| 682 | 465124 | 317214568 | 26·1151297 | 8·802272 |
| 683 | 466489 | 318611987 | 26·1342687 | 8·806572 |
| 684 | 467856 | 320013504 | 26·1533937 | 8·810868 |
| 685 | 469225 | 321419125 | 26·1725047 | 8·815159 |
| 686 | 470596 | 322828856 | 26·1916017 | 8·819447 |
| 687 | 471969 | 324242703 | 26·2106848 | 8·823730 |
| 688 | 473344 | 325660672 | 26·2297541 | 8·828009 |
| 689 | 474721 | 327082769 | 26·2488095 | 8·832285 |
| 690 | 476100 | 328509000 | 26·2678511 | 8·836556 |
| 691 | 477481 | 329939371 | 26·2868789 | 8·840822 |
| 692 | 478864 | 331373888 | 26·3058929 | 8·845085 |
| 693 | 480249 | 332812557 | 26·3248932 | 8·849344 |
| 694 | 481636 | 334255384 | 26·3438797 | 8·853598 |
| 695 | 483025 | 335702375 | 26·3628527 | 8·857849 |
| 696 | 484416 | 337153536 | 26·3818119 | 8·862095 |
| 697 | 485809 | 338608873 | 26·4007576 | 8·866337 |
| 698 | 487204 | 340068392 | 26·4196896 | 8·870575 |
| 699 | 488601 | 341532099 | 26·4386081 | 8·874809 |
| 700 | 490000 | 343000000 | 26·4575131 | 8·879040 |
| 701 | 491401 | 344472101 | 26·4764046 | 8·883266 |
| 702 | 492804 | 345948008 | 26·4952826 | 8·887488 |
| 703 | 494209 | 347428927 | 26·5141472 | 8·891706 |
| 704 | 495616 | 348913664 | 26·5329983 | 8·895920 |
| 705 | 497025 | 350402625 | 26·5518361 | 8·900130 |
| 706 | 498436 | 351895816 | 26·5706605 | 8·904336 |
| 707 | 499849 | 353393243 | 26·5894716 | 8·908538 |
| 708 | 501264 | 354894912 | 26·6082694 | 8·912736 |
| 709 | 502681 | 356400829 | 26·6270539 | 8·916931 |
| 710 | 504100 | 357911000 | 26·6458252 | 8·921121 |
| 711 | 505521 | 359425431 | 26·6645833 | 8·925307 |
| 712 | 506944 | 360944128 | 26·6833281 | 8·929490 |
| 713 | 508369 | 362467097 | 26·7020598 | 8·933668 |
| 714 | 509796 | 363994344 | 26·7207784 | 8·937843 |
| 715 | 511225 | 365525875 | 26·7394839 | 8·942014 |
| 716 | 512656 | 367061696 | 26·7581763 | 8·946180 |
| 717 | 514089 | 368601813 | 26·7768557 | 8·950343 |
| 718 | 515524 | 370146232 | 26·7955220 | 8·954502 |
| 719 | 516961 | 371694959 | 26·8141754 | 8·958658 |
| 720 | 518400 | 373248000 | 26·8328157 | 8·962809 |
| 721 | 519841 | 374805361 | 26·8514432 | 8·966957 |
| 722 | 521284 | 376367048 | 26·8700577 | 8·971100 |
| 723 | 522729 | 377933067 | 26·8886593 | 8·975240 |
| 724 | 524176 | 379503424 | 26·9072481 | 8·979376 |
| 725 | 525625 | 381078125 | 26·9258240 | 8·983508 |
| 726 | 527076 | 382657176 | 26·9443872 | 8·987637 |

| Number. | Square. | Cube. | Square Root. | Cube Root. |
|---|---|---|---|---|
| 727 | 528529 | 384240583 | 26·9629375 | 8·991762 |
| 728 | 529984 | 385828352 | 26·9814751 | 8·995883 |
| 729 | 531441 | 387420489 | 27·0000000 | 9·000000 |
| 730 | 532900 | 389017000 | 27·0185122 | 9·004113 |
| 731 | 534361 | 390617891 | 27·0370117 | 9·008222 |
| 732 | 535824 | 392223168 | 27·0554985 | 9·012328 |
| 733 | 537289 | 393832837 | 27·0739727 | 9·016430 |
| 734 | 538756 | 395446904 | 27·0924344 | 9·020529 |
| 735 | 540225 | 397065375 | 27·1108834 | 9·024623 |
| 736 | 541696 | 398688256 | 27·1293199 | 9·028714 |
| 737 | 543169 | 400315553 | 27·1477439 | 9·032802 |
| 738 | 544644 | 401947272 | 27·1661554 | 9·036885 |
| 739 | 546121 | 403583419 | 27·1845544 | 9·040965 |
| 740 | 547600 | 405224000 | 27·2029410 | 9·045041 |
| 741 | 549081 | 406869021 | 27·2213152 | 9·049114 |
| 742 | 550564 | 408518488 | 27·2396769 | 9·053183 |
| 743 | 552049 | 410172407 | 27·2580263 | 9·057248 |
| 744 | 553536 | 411830784 | 27·2763634 | 9·061309 |
| 745 | 555025 | 413493625 | 27·2946881 | 9·065367 |
| 746 | 556516 | 415160936 | 27·3130006 | 9·069422 |
| 747 | 558009 | 416832723 | 27·3313007 | 9·073472 |
| 748 | 559504 | 418508992 | 27·3495887 | 9·077519 |
| 749 | 561001 | 420189749 | 27·3678644 | 9·081563 |
| 750 | 562500 | 421875000 | 27·3861279 | 9·085603 |
| 751 | 564001 | 423564751 | 27·4043792 | 9·089639 |
| 752 | 565504 | 425259008 | 27·4226184 | 9·093672 |
| 753 | 567009 | 426957777 | 27·4408455 | 9·097701 |
| 754 | 568516 | 428661064 | 27·4590604 | 9·101726 |
| 755 | 570025 | 430368875 | 27·4772633 | 9·105748 |
| 756 | 571536 | 432081216 | 27·4954542 | 9·109766 |
| 757 | 573049 | 433798093 | 27·5136330 | 9·113781 |
| 758 | 574564 | 435519512 | 27·5317998 | 9·117793 |
| 759 | 576081 | 437245479 | 27·5499546 | 9·121801 |
| 760 | 577600 | 438976000 | 27·5680975 | 9·125805 |
| 761 | 579121 | 440711081 | 27·5862284 | 9·129806 |
| 762 | 580644 | 442450728 | 27·6043475 | 9·133803 |
| 763 | 582169 | 444194947 | 27·6224546 | 9·137797 |
| 764 | 583696 | 445943744 | 27·6405499 | 9·141788 |
| 765 | 585225 | 447697125 | 27·6586334 | 9·145774 |
| 766 | 586756 | 449455096 | 27·6767050 | 9·149757 |
| 767 | 588289 | 451217663 | 27·6947648 | 9·153737 |
| 768 | 589824 | 452984832 | 27·7128129 | 9·157713 |
| 769 | 591361 | 454756609 | 27·7308492 | 9·161686 |
| 770 | 592900 | 456533000 | 27·7488739 | 9·165656 |
| 771 | 594441 | 458314011 | 27·7668868 | 9·169622 |
| 772 | 595984 | 460099648 | 27·7848880 | 9·173585 |

| Number. | Square. | Cube. | Square Root. | Cube Root. |
|---|---|---|---|---|
| 773 | 597529 | 461889917 | 27·8028775 | 9·177544 |
| 774 | 599076 | 463684824 | 27·8208555 | 9·181500 |
| 775 | 600625 | 465484375 | 27·8388218 | 9·185452 |
| 776 | 602176 | 467288576 | 27·8567766 | 9·189401 |
| 777 | 603729 | 469097433 | 27·8747197 | 9·193347 |
| 778 | 605284 | 470910952 | 27·8926514 | 9·197289 |
| 779 | 606841 | 472729139 | 27·9105715 | 9·201228 |
| 780 | 608400 | 474552000 | 27·9284801 | 9·205164 |
| 781 | 609961 | 476379541 | 27·9463772 | 9·209096 |
| 782 | 611524 | 478211768 | 27·9642629 | 9·213025 |
| 783 | 613089 | 480048687 | 27·9821372 | 9·216950 |
| 784 | 614656 | 481890304 | 28·0000000 | 9·220872 |
| 785 | 616225 | 483736625 | 28·0178515 | 9·224791 |
| 786 | 617796 | 485587656 | 28·0356915 | 9·228706 |
| 787 | 619369 | 487443403 | 28·0535203 | 9·232618 |
| 788 | 620944 | 489303872 | 28·0713377 | 9·237527 |
| 789 | 622521 | 491169069 | 28·0891438 | 9·240433 |
| 790 | 624100 | 493039000 | 28·1069386 | 9·244335 |
| 791 | 625681 | 494913671 | 28·1247222 | 9·248234 |
| 792 | 627264 | 496793088 | 28·1424946 | 9·252130 |
| 793 | 628849 | 498677257 | 28·1602557 | 9·256022 |
| 794 | 630436 | 500566184 | 28·1780056 | 9·259911 |
| 795 | 632025 | 502459875 | 28·1957444 | 9·263797 |
| 796 | 633616 | 504358336 | 28·2134720 | 9·267679 |
| 797 | 635209 | 506261573 | 28·2311884 | 9·271559 |
| 798 | 636804 | 508169592 | 28·2488938 | 9·275435 |
| 799 | 638401 | 510082399 | 28·2665881 | 9·279308 |
| 800 | 640000 | 512000000 | 28·2842712 | 9·283177 |
| 801 | 641601 | 513922401 | 28·3019434 | 9·287044 |
| 802 | 643204 | 515849608 | 28·3196045 | 9·290907 |
| 803 | 644809 | 517781627 | 28·3372546 | 9·294767 |
| 804 | 646416 | 519718464 | 28·3548938 | 9·298623 |
| 805 | 648025 | 521660125 | 28·3725219 | 9·302477 |
| 806 | 649636 | 523606616 | 28·3901391 | 9·306327 |
| 807 | 651249 | 525557943 | 28·4077454 | 9·310175 |
| 808 | 652864 | 527514112 | 28·4253408 | 9·314019 |
| 809 | 654481 | 529475129 | 28·4429253 | 9·317859 |
| 810 | 656100 | 531441000 | 28·4604989 | 9·321697 |
| 811 | 657721 | 533411731 | 28·4780617 | 9·325532 |
| 812 | 659344 | 535387328 | 28·4956137 | 9·329363 |
| 813 | 660969 | 537366797 | 28·5131549 | 9·333191 |
| 814 | 662596 | 539353144 | 28·5306852 | 9·337016 |
| 815 | 664225 | 541343375 | 28·5482048 | 9·340838 |
| 816 | 665856 | 543338496 | 28·5657137 | 9·344657 |
| 817 | 667489 | 545338513 | 28·5832119 | 9·348473 |
| 818 | 669124 | 547343432 | 28·6006993 | 9·352285 |

| Number. | Square. | Cube. | Square Root. | Cube Root. |
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| 819 | 670761 | 549353259 | 28·6181760 | 9·356095 |
| 820 | 672400 | 551368000 | 28·6356421 | 9·359901 |
| 821 | 674041 | 553387661 | 28·6530976 | 9·363704 |
| 822 | 675684 | 555412248 | 28·6705424 | 9·367505 |
| 823 | 677329 | 557441767 | 28·6879766 | 9·371302 |
| 824 | 678976 | 559476224 | 28·7054002 | 9·375096 |
| 825 | 680625 | 561515625 | 28·7228132 | 9·378887 |
| 826 | 682276 | 563559976 | 28·7402157 | 9·382675 |
| 827 | 683929 | 565609283 | 28·7576077 | 9·386460 |
| 828 | 685584 | 567663552 | 28·7749891 | 9·390241 |
| 829 | 687241 | 569722789 | 28·7923601 | 9·394020 |
| 830 | 688900 | 571787000 | 28·8097206 | 9·397796 |
| 831 | 690561 | 573856191 | 28·8270706 | 9·401569 |
| 832 | 692224 | 575930368 | 28·8444102 | 9·405338 |
| 833 | 693889 | 578009537 | 28·8617394 | 9·409105 |
| 834 | 695556 | 580093704 | 28·8790582 | 9·412869 |
| 835 | 697225 | 582182875 | 28·8963666 | 9·416630 |
| 836 | 698896 | 584277056 | 28·9136646 | 9·420387 |
| 837 | 700569 | 586376253 | 28·9309523 | 9·424141 |
| 838 | 702244 | 588480472 | 28·9482297 | 9·427893 |
| 839 | 703921 | 590589719 | 28·9654967 | 9·431642 |
| 840 | 705600 | 592704000 | 28·9827535 | 9·435388 |
| 841 | 707281 | 594823321 | 29·0000000 | 9·439130 |
| 842 | 708964 | 596947688 | 29·0172363 | 9·442870 |
| 843 | 710649 | 599077107 | 29·0344623 | 9·446607 |
| 844 | 712336 | 601211584 | 29·0516781 | 9·450341 |
| 845 | 714025 | 603351125 | 29·0688837 | 9·454071 |
| 846 | 715716 | 605495736 | 29·0860791 | 9·457799 |
| 847 | 717409 | 607645423 | 29·1032644 | 9·461524 |
| 848 | 719104 | 609800192 | 29·1204396 | 9·465247 |
| 849 | 720801 | 611960049 | 29·1376046 | 9·468966 |
| 850 | 722500 | 614125000 | 29·1547595 | 9·472682 |
| 851 | 724201 | 616295051 | 29·1719043 | 9·476395 |
| 852 | 725904 | 618470208 | 29·1890390 | 9·480106 |
| 853 | 727609 | 620650477 | 29·2061637 | 9·483813 |
| 854 | 729316 | 622835864 | 29·2232784 | 9·487518 |
| 855 | 731025 | 625026375 | 29·2403830 | 9·491219 |
| 856 | 732736 | 627222016 | 29·2574777 | 9·494918 |
| 857 | 734449 | 629422793 | 29·2745623 | 9·498614 |
| 858 | 736164 | 631628712 | 29·2916370 | 9·502307 |
| 859 | 737881 | 633839779 | 29·3087018 | 9·505998 |
| 860 | 739600 | 636056000 | 29·3257566 | 9·509685 |
| 861 | 741321 | 638277381 | 29·3428015 | 9·513369 |
| 862 | 743044 | 640503928 | 29·3598365 | 9·517051 |
| 863 | 744769 | 642735647 | 29·3768616 | 9·520730 |
| 864 | 746496 | 644972544 | 29·3938769 | 9·524406 |

| Num-ber. | Square. | Cube. | Square Root. | Cube Root. |
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| 865 | 748225 | 647214625 | 29·4108823 | 9·528079 |
| 866 | 749956 | 649461896 | 29·4278779 | 9·531749 |
| 867 | 751689 | 651714363 | 29·4448637 | 9·535417 |
| 868 | 753424 | 653972032 | 29·4618397 | 9·539081 |
| 869 | 755161 | 656234909 | 29·4788059 | 9·542743 |
| 870 | 756900 | 658503000 | 29·4957624 | 9·546402 |
| 871 | 758641 | 660776311 | 29·5127091 | 9·550058 |
| 872 | 760384 | 663054848 | 29·5296461 | 9·553712 |
| 873 | 762129 | 665338617 | 29·5465734 | 9·557363 |
| 874 | 763876 | 667627624 | 29·5634910 | 9·561010 |
| 875 | 765625 | 669921875 | 29·5803989 | 9·564655 |
| 876 | 767376 | 672221376 | 29·5972972 | 9·568297 |
| 877 | 769129 | 674526133 | 29·6141858 | 9·571937 |
| 878 | 770884 | 676836152 | 29·6310648 | 9·575574 |
| 879 | 772641 | 679151439 | 29·6479325 | 9·579208 |
| 880 | 774400 | 681472000 | 29·6647939 | 9·582839 |
| 881 | 776161 | 683797841 | 29·6816442 | 9·586468 |
| 882 | 777924 | 686128968 | 29·6984848 | 9·590093 |
| 883 | 779689 | 688465387 | 29·7153159 | 9·593716 |
| 884 | 781456 | 690807104 | 29·7321375 | 9·597337 |
| 885 | 783225 | 693154125 | 29·7489496 | 9·600954 |
| 886 | 784996 | 695506456 | 29·7657521 | 9·604569 |
| 887 | 786769 | 697864103 | 29·7825452 | 9·608181 |
| 888 | 788544 | 700227072 | 29·7993289 | 9·611791 |
| 889 | 790321 | 702595369 | 29·8161030 | 9·615397 |
| 890 | 792100 | 704969000 | 29·8328678 | 9·619001 |
| 891 | 793881 | 707347971 | 29·8496231 | 9·622603 |
| 892 | 795664 | 709732288 | 29·8663690 | 9·626201 |
| 893 | 797449 | 712121957 | 29·8831056 | 9·629797 |
| 894 | 799236 | 714516984 | 29·8998328 | 9·633390 |
| 895 | 801025 | 716917375 | 29·9165506 | 9·636981 |
| 896 | 802816 | 719323136 | 29·9332591 | 9·640569 |
| 897 | 804609 | 721734273 | 29·9499583 | 9·644154 |
| 898 | 806404 | 724150792 | 29·9666481 | 9·647736 |
| 899 | 808201 | 726572699 | 29·9833287 | 9·651316 |
| 900 | 810000 | 729000000 | 30·0000000 | 9·654893 |
| 901 | 811801 | 731432701 | 30·0166620 | 9·658468 |
| 902 | 813604 | 733870808 | 30·0333148 | 9·662040 |
| 903 | 815409 | 736314327 | 30·0499584 | 9·665609 |
| 904 | 817216 | 738763264 | 30·0665928 | 9·669176 |
| 905 | 819025 | 741217625 | 30·0832179 | 9·672740 |
| 906 | 820836 | 743677416 | 30·0998339 | 9·676301 |
| 907 | 822649 | 746142643 | 30·1164407 | 9·679860 |
| 908 | 824464 | 748613312 | 30·1330383 | 9·683416 |
| 909 | 826281 | 751089429 | 30·1496269 | 9·686970 |
| 910 | 828100 | 753571000 | 30·1662063 | 9·690521 |

| Number. | Square. | Cube. | Square Root. | Cube Root. |
|---|---|---|---|---|
| 911 | 829921 | 756058031 | 30·1827765 | 9·694069 |
| 912 | 831744 | 758550528 | 30·1993377 | 9·697615 |
| 913 | 833569 | 761048497 | 30·2158899 | 9·701158 |
| 914 | 835396 | 763551944 | 30·2324329 | 9·704698 |
| 915 | 837225 | 766060875 | 30·2489669 | 9·708236 |
| 916 | 839056 | 768575296 | 30·2654919 | 9·711772 |
| 917 | 840889 | 771095213 | ·30·2820079 | 9·715305 |
| 918 | 842724 | 773620632 | 30·2985148 | 9·718835 |
| 919 | 844561 | 776151559 | 30·3150128 | 9·722363 |
| 920 | 846400 | 778688000 | 30·3315018 | 9·725888 |
| 921 | 848241 | 781229961 | 30·3479818 | 9·729410 |
| 922 | 850084 | 783777448 | 30·3644529 | 9·732930 |
| 923 | 851929 | 786330467 | 30·3809151 | 9·736448 |
| 924 | 853776 | 788889024 | 30·3973683 | 9·739963 |
| 925 | 855625 | 791453125 | 30·4138127 | 9·743475 |
| 926 | 857476 | 794022776 | 30·4302481 | 9·746985 |
| 927 | 859329 | 796597983 | 30·4166747 | 9·750493 |
| 928 | 861184 | 799178752 | ·30·4530924 | 9·753998 |
| 929 | 863041 | 801765089 | 30·4795013 | ·9·757500 |
| 930 | 864900 | 804357000 | 30·4959014 | 9·761000 |
| 931 | 866761 | 806954491 | 30·5122926 | 9·764497 |
| 932 | 868624 | 809557568 | 30·5286750 | 9·767992 |
| 933 | 870489 | 812166237 | 30·5450487 | 9·771484 |
| 934 | 872356 | 814780504 | 30·5614136 | 9·774974 |
| 935 | 874225 | 817400375 | 30·5777697 | 9·778461 |
| 936 | 876096 | 820025856 | 30·5941171 | 9·782946 |
| 937 | 877969 | 822656953 | 30·6104557 | 9·785428 |
| 938 | 879844 | 825293672 | 30·6267857 | 9·788908 |
| 939 | 881721 | 827936019 | 30·6431069 | 9·792386 |
| 940 | 883600 | 830584000 | 30·6594194 | 9·795861 |
| 941 | 885481 | 833237621 | 30·6757233 | 9·799333 |
| 942 | 887364 | 835896888 | 30·6920185 | 9·802803 |
| 943 | 889249 | 838561807 | 30·7083051 | 9·806271 |
| 944 | 891136 | 841232384 | 30·7245830 | 9·809736 |
| 945 | 893025 | 843908625 | 30·7408523 | 9·813198 |
| 946 | 894916 | 846590536 | 30·7571130 | 9·816659 |
| 947 | 896809 | 849278123 | 30·7733651 | 9·820117 |
| 948 | 898704 | 851971392 | 30·7896086 | 9·823572 |
| 949 | 900601 | 854670349 | 30·8058436 | 9·827025 |
| 950 | 902500 | 857375000 | 30·8220700 | 9·830475 |
| 951 | 904401 | 860085351 | 30·8382879 | 9·833923 |
| 952 | 906304 | 862801408 | 30·8544972 | 9·837369 |
| 953 | 908209 | 865523177 | 30·8706981 | 9·840812 |
| 954 | 910116 | 868250664 | 30·8868904 | 9·844253 |
| 955 | 912025 | 870983875 | 30·9030743 | 9·847692 |
| 956 | 913936 | 873722816 | 30·9192497 | 9·851128 |

| Number. | Square. | Cube. | Square Root. | Cube Root. |
|---|---|---|---|---|
| 957 | 915849 | 876467493 | 30·9354166 | 9·854561 |
| 958 | 917764 | 879217912 | 30·9515751 | 9·857992 |
| 959 | 919681 | 881974079 | 30·9677251 | 9·861421 |
| 960 | 921600 | 884736000 | 30·9838668 | 9·864848 |
| 961 | 923521 | 887503681 | 31·0000000 | 9·868272 |
| 962 | 925444 | 890277128 | 31·0161248 | 9·871694 |
| 963 | 927369 | 893056347 | 31·0322413 | 9·875113 |
| 964 | 929296 | 895841344 | 31·0483494 | 9·878530 |
| 965 | 931225 | 898632125 | 31·0644491 | 9·881945 |
| 966 | 933156 | 901428696 | 31·0805405 | 9·885357 |
| 967 | 935089 | 904231063 | 31·0966236 | 9·888767 |
| 968 | 937024 | 907039232 | 31·1126984 | 9·892174 |
| 969 | 938961 | 909853209 | 31·1287648 | 9·895580 |
| 970 | 940900 | 912673000 | 31·1448230 | 9·898983 |
| 971 | 942841 | 915498611 | 31·1608729 | 9·902383 |
| 972 | 944784 | 918330048 | 31·1769145 | 9·905781 |
| 973 | 946729 | 921167317 | 31·1929479 | 9·909177 |
| 974 | 948676 | 924010424 | 31·2089731 | 9·912571 |
| 975 | 950625 | 926859375 | 31·2249900 | 9·915962 |
| 976 | 952576 | 929714176 | 31·2409987 | 9·919351 |
| 977 | 954529 | 932574833 | 31·2569992 | 9·922738 |
| 978 | 956484 | 935441352 | 31·2729915 | 9·926122 |
| 979 | 958441 | 938313739 | 31·2889757 | 9·929504 |
| 980 | 960400 | 941192001 | 31·3049517 | 9·932883 |
| 981 | 962361 | 944076141 | 31·3209195 | 9·936261 |
| 982 | 964324 | 946966168 | 31·3368792 | 9·939636 |
| 983 | 966289 | 949862087 | 31·3528308 | 9·943009 |
| 984 | 968256 | 952763904 | 31·3687743 | 9·946379 |
| 985 | 970225 | 955671625 | 31·3847097 | 9·949747 |
| 986 | 972196 | 958585256 | 31·4006369 | 9·953113 |
| 987 | 974169 | 961504803 | 31·4165561 | 9·956477 |
| 988 | 976144 | 964430272 | 31·4324673 | 9·959839 |
| 989 | 978121 | 967361669 | 31·4483704 | 9·963198 |
| 990 | 980100 | 970299000 | 31·4642654 | 9·966554 |
| 991 | 982081 | 973242271 | 31·4801525 | 9·969909 |
| 992 | 984064 | 976191488 | 31·4960315 | 9·973262 |
| 993 | 986049 | 979146657 | 31·5119025 | 9·976612 |
| 994 | 988036 | 982107784 | 31·5277655 | 9·979959 |
| 995 | 990025 | 985074875 | 31·5436206 | 9·983304 |
| 996 | 992016 | 988047936 | 31·5594677 | 9·986648 |
| 997 | 994009 | 991026973 | 31·5753068 | 9·989990 |
| 998 | 996004 | 994011992 | 31·5911380 | 9·993328 |
| 999 | 998001 | 997002999 | 31·6069613 | 9·996665 |
| 1000 | 1000000 | 1000000000 | 31·6227766 | 10·000000 |
| 1001 | 1002001 | 1003003001 | 31·6385840 | 10·003332 |
| 1002 | 1004004 | 1006012008 | 31·6543836 | 10·006662 |

## Of Ratios, Proportions and Progressions.

Numbers are compared to each other in two different ways: the one comparison confiders the difference of the two numbers, and is named Arithmetical Relation; and the difference sometimes the Arithmetical Ratio: the other confiders their quotient, which is called Geometrical Relation, and the quotient the Geometrical Ratio. So, of thefe two numbers 6 and 3, the difference, or arithmetical ratio, is $6 - 3$ or 3; but the geometrical ratio is $\frac{6}{3}$ or 2.

There muft be two numbers to form a comparifon: the number which is compared, being placed firft, is called the Antecedent; and that to which it is compared, the Confequent. So, in the two numbers above, 6 is the antecedent, and 3 the confequent.

If two or more couplets of numbers have equal ratios, or equal differences, the equality is named Proportion, and the terms of the ratios Proportionals. So, the two couplets, 4, 2 and 8, 6, are arithmetical proportionals, becaufe $4 - 2 = 8 - 6 = 2$; and the two couplets 4, 2 and 6, 3, are geometrical proportionals, becaufe $\frac{4}{2} = \frac{6}{3} = 2$, the fame ratio.

To denote numbers as being geometrically proportional, a colon is fet between the terms of each couplet, to denote their ratio; and a double colon, or elfe a mark of equality, between the couplets or ratios. So, the four proportionals, 4, 2, 6, 3 are fet thus, $4 : 2 :: 6 : 3$, which means, that 4 is to 2 as 6 is to 3; or thus, $4 : 2 = 6 : 3$; or thus, $\frac{4}{2} = \frac{6}{3}$, both which mean, that the ratio of 4 to 2, is equal to the ratio of 6 to 3.

Proportion is diftinguifhed into Continued and Difconti-nued. When the difference or ratio of the confequent of one couplet and the antecedent of the next couplet, is not the fame as the common difference or ratio of the couplets, the proportion is difcontinued. So, 4, 2, 8, 6 are in difcontinued arithmetical proportion, becaufe $4 - 2 = 8 - 6 = 2$, where-as $8 - 2 = 6$: and 4, 2, 6, 3 are in difcontinued geometrical proportion, becaufe $\frac{4}{2} = \frac{6}{3} = 2$, but $\frac{6}{2} = 3$, which is not the fame.

But when the difference or ratio of every two fucceeding terms is the fame quantity, the proportion is faid to be Con-tinued, and the numbers themfelves a feries of Continued

Propor-

Proportionals, or a Progreſſion. So 2, 4, 6, 8 form an arithmetical progreſſion, becauſe $4 - 2 = 6 - 4 = 8 - 6 = 2$, all the ſame common difference; and 2, 4, 8, 16 a geometrical progreſſion, becauſe $\frac{4}{2} = \frac{8}{4} = \frac{16}{8} = 2$, all the ſame ratio.

When the following terms of a Progreſſion increaſe, or exceed each other, it is called an Aſcending Progreſſion or Series; but when the terms decreaſe, it is a Deſcending one.

So, 0, 1, 2, 3, 4, &c, is an aſcending arithmetical progreſſion, but 9, 7, 5, 3, 1, &c, is a deſcending arithmetical progreſſion, Alſo 1, 2, 4, 8, 16, &c, is an aſcending geometrical progreſſion, and 16, 8, 4, 2, 1, &c, is a deſcending geometrical progreſſion,

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## ARITHMETICAL PROPORTION AND PROGRESSION.

THE firſt and laſt terms of a Progreſſion, are called the Extremes; and the other terms, lying between them, the Means.

The moſt uſeful part of arithmetical proportions, is contained in the following theorems:

THEOREM 1. If four quantities be in arithmetical proportion, the ſum of the two extremes will be equal to the ſum of the two means.

Thus, of the four 2, 4, 6, 8, here $2 + 8 = 4 + 6 = 10$.

THEOREM 2. In any continued arithmetical progreſſion, the ſum of the two extremes is equal to the ſum of any two means that are equally diſtant from them, or equal to double the middle term when there is an uneven number of terms.

Thus, in the terms 1, 3, 5, it is $1 + 5 = 3 + 3 = 6$.

And in the ſeries 2, 4, 6, 8, 10, 12, 14, it is $2 + 14 = 4 + 12 = 6 + 10 = 8 + 8 = 16$.

THEOREM 3. The difference between the extreme terms of an arithmetical progreſſion, is equal to the common difference of the ſeries multiplied by one leſs than the number of the terms.

So, of the ten terms, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, the common difference is 2, and one leſs than the number of terms 9; then the difference of the extremes is $20 - 2 = 18$, and $2 \times 9 = 18$ alſo.

Confequently, the greateſt term is equal to the leaſt term added to the product of the common difference multiplied by 1 lefs than the number of terms.

THEOREM 4. The fum of all the terms, of any arithmetical progreſſion, is equal to the fum of the two extremes multiplied by the number of terms, and divided by 2; or the fum of the two extremes multiplied by the number of the terms, gives double the fum of all the terms in the feries.

This is made evident by fetting the terms of the feries in an inverted order, under the fame feries in a direct order, and adding the correfponding terms together in that order. Thus,

$$\begin{array}{lcccccccc}
\text{in the feries} & 1, & 3, & 5, & 7, & 9, & 11, & 13, & 15; \\
\text{ditto inverted} & 15, & 13, & 11, & 9, & 7, & 5, & 3, & 1; \\
\text{the fums are} & 16 + & 16 + & 16 + & 16 + & 16 + & 16 + & 16 + & 16,
\end{array}$$

which muſt be double the fum of the fingle feries, and is equal to the fum of the extremes repeated fo often as are the number of the terms.

From thefe theorems may readily be found any one of thefe five parts; the two extremes, the number of terms, the common difference, and the fum of all the terms, when any three of them are given; as in the following Problems:

## PROBLEM I.

*Given the Extremes, and the Number of Terms; to find the Sum of all the Terms.*

### RULE.

ADD the extremes together, multiply the fum by the number of terms, and divide by 2.

### EXAMPLES.

1. The extremes being 3 and 19, and the number of terms 9; required the fum of the terms?

$$\begin{array}{r}
19 \\
3 \\
\hline
22 \\
9 \\
\hline
2\,)\,198 \\
\hline
\text{Anf.}\ 99
\end{array}$$

Or $\dfrac{19 + 3}{2} \times 9 = \dfrac{22}{2} \times 9 = 11 \times 9 = 99.$

2. It is required to find the number of all the ſtrokes a clock ſtrikes in one whole revolution of the index, or in 12 hours?　　　　　　　　　　　　　　　　　　　　Anf. 78.

Ex.

Ex. 3. How many ſtrokes do the clocks of Venice ſtrike in the compaſs of the day, which go right on from 1 to 24 o'clock ?                                                 Anſ. 300.

4. What debt can be diſcharged in a year, by weekly payments in arithmetical progreſſion, the firſt payment being 1s, and the laſt or 52d payment 5l 3s ?          Anſ. 135l 4s.

### PROBLEM II.

*Given the Extremes, and the Number of Terms; to find the Common Difference.*

#### R U L E.

Subtract the leſs extreme from the greater, and divide the remainder by 1 leſs than the number of terms, for the common difference.

#### E X A M P L E S.

1. The extremes being 3 and 19, and the number of terms 9 ; required the common difference ?

$$19$$
$$\underline{\quad 3\quad}$$
$$8\,)\,16$$
$$\overline{\text{Anſ.}\quad 2}$$

Or $\dfrac{19-3}{9-1} = \dfrac{16}{8} = 2.$

2. If the extremes be 10 and 70, and the number of terms 21 ; what is the common difference, and the ſum of the ſeries ?                Anſ. the com. diff. is 3, and the ſum is 840.

3. A certain debt can be diſcharged in one year, by weekly payments in arithmetical progreſſion, the firſt payment being 1s, and the laſt 5l 3s ; what is the common difference of the terms ?                                  Anſ. 2.

### PROBLEM III.

*Given one of the Extremes, the Common Difference, and the Number of Terms; to find the other Extreme, and the Sum of the Series.*

#### R U L E.

Multiply the common difference by 1 leſs than the number of terms, and the product will be the difference of the extremes : Therefore add the product to the leſs extreme, to give the greater ; or ſubtract it from the greater, to give the leſs.

E X A M-

1. Given the leaſt term 3, the common difference 2, of an arithmetical ſeries of 9 terms; to find the greateſt term, and the ſum of the ſeries ?

$$
\begin{array}{r}
2 \\
8 \\
\hline
16 \\
3 \\
\hline
\end{array}
$$

19 the greateſt term
3 the leaſt
22 ſum
9 number of terms.
2 ) 198
99 the ſum of the ſeries.

2. If the greateſt term be 70, the common difference 3, and the number of terms 21 ; what is the leaſt term and the ſum of the ſeries ?

Anſ. The leaſt term is 10, and the ſum is 840.

3. A debt can be diſcharged in a year, by paying 1 ſhilling the firſt week, 3 ſhillings the ſecond, and ſo on, always 2 ſhillings more every week; what is the debt, and what will the laſt payment be ?

Anſ. The laſt payment will be 5l 3s, and the debt is 135l 4s.

### PROBLEM IV.

*To find an Arithmetical Mean Proportional between Two Given Terms.*

### RULE.

ADD the two given extremes or terms together, and take half their ſum for the arithmetical mean required. Or, ſub-tract the leſs extreme from the greater, and half the remain-der will be the common difference ; which being added to the leſs extreme, or ſubtracted from the greater, will give the mean required.

### EXAMPLE.

To find an arithmetical mean between the two numbers 4 and 14.

Here

Here   Or 14    Or 14
14      4       5
 4    2 ) 10      9
2 ) 18     5 the com dif.
Anf. 9     4 the lefs extreme.
        9

So that 9 is the mean required, by both methods.

## PROBLEM V.

*To find Two Arithmetical Means between Two Given Extremes.*

### RULE.

SUBTRACT the lefs extreme from the greater, and divide the difference by 3, fo will the quotient be the common difference ; which being continually added to the lefs extreme, or taken from the greater, gives the means.

### EXAMPLE.

To find two arithmetical means between 2 and 8.

Here 8
  2
3 ) 6   Then 2 + 2 = 4 the one mean,
com. dif. 2   and 4 + 2 = 6 the other mean.

## PROBLEM VI.

*To find any Number of Arithmetical Means between Two Given Terms or Extremes.*

### RULE.

SUBTRACT the lefs extreme from the greater, and divide the difference by 1 more than the number of means required to be found, which will give the common difference ; then this being added continually to the leaft term, or fubtracted from the greateft, will give the mean terms required.

EXAM-

### EXAMPLE.

To find five arithmetical means between 2 and 14.

Here 14
        2
6 ) 12     Then by adding this com. dif. continually,
com. dif. 2       the means are found 4, 6, 8, 10, 12.

See more of Arithmetical Progreſſion in the Algebra.

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## GEOMETRICAL PROPORTION and PROGRESSION.

The moſt uſeful part of Geometrical Proportion, is contained in the following theorems.

Theorem 1. If four quantities be in geometrical proportion, the product of the two extremes will be equal to the product of the two means.

Thus, in the four 2, 4, 3, 6, it is $2 \times 6 = 3 \times 4 = 12$.

And hence, if the product of the two means be divided by one of the extremes, the quotient will give the other extreme. So, of the above numbers, the product of the means $12 \div 2 = 6$ the one extreme, and $12 \div 6 = 2$ the other extreme; and this is the foundation and reaſon of the practice in the Rule-of-Three.

Theorem 2. In any continued geometrical progreſſion, the product of the two extremes is equal to the product of any two means that are equally diſtant from them, or equal to the ſquare of the middle term when there is an uneven number of terms.

Thus, in the terms 2, 4, 8, it is $2 \times 8 = 4 \times 4 = 16$.
And in the ſeries 2, 4, 8, 16, 32, 64, 128,
    it is $2 \times 128 = 4 \times 64 = 8 \times 32 = 16 \times 16 = 256$.

Theorem 3. The quotient of the extreme terms of a geometrical progreſſion, is equal to the common ratio of the ſeries raiſed to the power denoted by 1 leſs than the number of the terms.

Conſequently the greateſt term is equal to the leaſt term multiplied by the ſaid quotient.

So,

So, of the ten terms 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, the common ratio is 2, one lefs than the number of terms 9 ; then the quotient of the extremes is $\dfrac{1024}{2} = 512$, and $2^9 = 512$ alfo.

THEOREM 4. The fum of all the terms, of any geometrical progreffion, is found by adding the greateft term to the difference of the extremes divided by 1 lefs than the ratio.

So, the fum of 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, (whofe ratio is 2), is $1024 + \dfrac{1024 - 2}{2 - 1} = 1024 + 1022 = 2046.$

The foregoing, and feveral other properties of geometrical proportion, are demonftrated more at large in the Algebraic part of this work. A few examples may here be added of the theorems, juft delivered, with fome problems concerning mean proportionals.

### EXAMPLES.

1. The leaft of ten terms, in geometrical progreffion, being 1, and the ratio 2 ; what is the greateft term, and the fum of all the terms ?

Anf. The greateft term is 512, and the fum 1023.

2. What debt may be difcharged in a year, or 12 months, by paying 1l the firft month, 2l the fecond, 4l the third, and fo on, each fucceeding payment being double the laft ; and what will the laft payment be ?

Anf. The debt 4095l, and the laft payment 2048l.

### PROBLEM I.

*To find One Mean Geometrical Proportional between any Two Numbers.*

### RULE.

MULTIPLY the two numbers together, and extract the fquare root of the product, which will give the mean proportional fought.

*Or*, Divide the greater term by the lefs, and extract the fquare-root of the quotient, which will give the common ratio of the three terms : then multiply the lefs term by the ratio, or divide the greater term by it, either of thefe will give the middle term required.

E X A M-

### EXAMPLE.

To find a geometrical mean between the two numbers 3 and 12.

*Firſt Way.*       *Second Way.*

  12      3 ) 12 ( 4, its root is 2 the ratio.

  <u> 3</u>

  36 ( 6 the mean.   Then 3 × 2 = 6 the mean,

  <u>36</u>       Or   12 ÷ 2 = 6 ditto.

### PROBLEM II.

*To find Two Geometrical Mean Proportionals between any Two Numbers.*

### RULE.

DIVIDE the greater number by the leſs, and extract the cube root of the quotient, which will give the common ratio of the terms. Then multiply the leaſt given term by the ratio for the firſt mean, and this mean again by the ratio for the ſecond mean: or, divide the greater of the two given terms by the ratio for the greater mean, and divide this again by the ratio for the leſs mean.

### EXAMPLE.

To find two geometrical mean proportionals between 3 and 24.

Here   3 ) 24 ( 8 ; its cube root 2 is the ratio.

Then   3 × 2 =   6, and 6 × 2 = 12, the two means,

Or    24 ÷ 2 = 12, and 12 ÷ 2 =   6, the ſame.

That is, the two means between 3 and 24, are 6 and 12.

### PROBLEM III.

*To find any Number of Geometrical Mean Proportionals between Two Numbers.*

### RULE.

DIVIDE the greater number by the leſs, and extract ſuch root of the quotient whoſe index is 1 more than the number of means required, that is, the 2d root for one mean, the 3d root for two means, the 4th root for three means, and ſo on ; and that root will be the common ratio of all the terms. Then, with the ratio multiply continually from the firſt term, or divide continually from the laſt or greateſt term.

## EXAMPLE.

To find four geometrical mean proportionals between 3 and 96

Here 3 ) 96 ( 32; the 5th root of which is 2, the ratio. Then $3 \times 2 = 6$, and $6 \times 2 = 12$, and $12 \times 2 = 24$, and $24 \times 2 = 48$. Or $96 \div 2 = 48$, and $48 \div 2 = 24$, and $24 \div 2 = 12$, and $12 \div 2 = 6$.

That is 6, 12, 24, 48 are the four means between 3 and 96.

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## OF MUSICAL PROPORTION.

THERE is also a third kind of proportion, called Musical, which being but of little or no common use, a very short account of it may here suffice.

Musical Proportion is when, of three numbers, the first has the same proportion to the third, as the difference between the first and second, hath to the difference between the second and third.

As in these three, 6, 8, 12;
Where $6 : 12 :: 8 - 6 : 12 - 8$,
that is $6 : 12 :: 2 : 4$.

When four numbers are in musical proportion; then the first has the same proportion to the fourth, as the difference between the first and second hath to the difference between the third and fourth.

As in these, 6, 8, 12, 18;
where $6 : 18 :: 8 - 6 : 18 - 12$,
that is $6 : 18 :: 2 : 6$.

When numbers are in musical progression, their reciprocals are in arithmetical progression; and the converse, that is, when numbers are in arithmetical progression, their reciprocals are in musical progression.

So in these musicals 6, 8, 12, their reciprocals, $\frac{1}{6}$, $\frac{1}{8}$, $\frac{1}{12}$, are in arithmetical progression; for $\frac{1}{6} + \frac{1}{12} = \frac{3}{12} = \frac{1}{4}$; and $\frac{1}{8} + \frac{1}{8} = \frac{2}{8} = \frac{1}{4}$; that is, the sum of the extremes is equal to double the mean, which is the property of arithmeticals.

The method of finding out numbers in musical proportion, is best expressed by letters in Algebra.

FELLOW-

## FELLOWSHIP, or PARTNERSHIP.

FELLOWSHIP is a rule, by which any sum or quantity may be divided into any number of parts, which shall be in any given proportion to one another.

By this rule are adjusted the gains or loss or charges of partners in company: or the effects of bankrupts, or legacies in case of a deficiency of assets or effects; or the shares of prizes, or the numbers of men to form certain detachments; or the division of waste lands among a number of proprietors.

Fellowship is either Single or Double. It is Single, when the shares or portions are to be proportional each to one single given number only; as when the stocks of partners are all employed for the same time: And Double, when each portion is to be proportional to two or more numbers; as when the stocks of partners are employed for different times.

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## SINGLE FELLOWSHIP.

### GENERAL RULE.

ADD together the numbers that denote the proportion of the shares. Then say,

As the sum of the said proportional numbers,
Is to the whole sum to be parted or divided,
So is each several proportional number,
To the corresponding share or part.

Or, As the whole stock, is to the whole gain or loss,
So is each man's particular stock,
To his particular share of the gain or loss.

To PROVE THE WORK. Add all the shares or parts together, and the sum will be equal to the whole number to be shared, when the work is right.

### EXAMPLES.

1. To divide the number 240 into three such parts, as shall be in proportion to each other as the three numbers 1, 2, and 3.

Here

Here $1 + 2 + 3 = 6$ the fum of the numbers.

Then, as 6 : 240 :: 1 : 40 the 1ſt part,
and as 6 : 240 :: 2 : 80 the 2d part,
alſo as 6 : 240 :: 3 : 120 the 3d part.
Sum of all 240, the proof.

2. Three perſons, A, B, C, freighted a ſhip with 340 tuns of wine ; of which, A loaded 110 tuns, B 97, and C the reſt : in a ſtorm the ſeamen were obliged to throw overboard 85 tuns ; how much muſt each perſon ſuſtain of the loſs ?

Here 110 + 97 = 207 tuns, loaded by A and B
theref. 340 — 207 = 133 tuns, loaded by C.

Hence, as 340 : 85 :: 110
or as 4 : 1 :: 110 : $27\frac{1}{2}$ tuns = A's loſs ;
and as 4 : 1 :: 97 : $24\frac{1}{4}$ tuns = B's loſs ;
alſo as 4 : 1 :: 133 : $33\frac{1}{4}$ tuns = C's loſs.

Sum 85 tuns, the proof.

3. Two merchants, C and D, made a ſtock of 120l ; of which C contributed 75l, and D the reſt ; by trading they gained 30l ; what muſt each have of it ?

Anſ. C 18l 15s, and D 11l 5s.

4. Three merchants, E, F, G, make a ſtock of 700l, of which E contributed 123l, F 358l, and G the reſt ; by trading they gain 125l 10s ; what muſt each have of it ?

Anſ. E muſt have 22l 1s 0d $2\frac{2}{35}$q.
F — — 64 3 8 $0\frac{32}{35}$.
G — — 39 5 3 $1\frac{1}{35}$.

5. A General impoſing a contribution * of 700l. on four villages, to be paid in proportion to the number of inhabitants contained in each ; the 1ſt containing 250, the 2d 350, the 3d 400, and the 4th 500 perſons ; what part muſt each village pay ?

Anſ. the 1ſt to pay 116l 13s 4d.
the 2d — — 163 6 8
the 3d — — 186 13 4
the 4th — — 233 6 8

* Contribution is a tax paid by provinces, towns, villages, &c. to excuſe them from being plundered. It is paid in proviſions or in money, and ſometimes in both.

Ex. 6.

Ex. 6. A piece of ground, confifting of 37ac 2ro 14ps, is to be divided among three perfons, L, M, and N, in proportion to their eftates : now if L's eftate be worth 500l a year, M's 320l, and N's 75l; what quantity of land muft each one have ?          Anf. L muft have 20ac 3ro 39$\frac{139}{179}$ps.

$$M \quad - \quad 13 \quad 1 \quad 30\tfrac{46}{179}$$
$$N \quad - \quad 3 \quad 0 \quad 23\tfrac{173}{179}.$$

7. A perfon is indebted to O 57l 15s, to P 108l 3s 8d, to Q 22l 10d, and to R 73l; but at his deceafe, his effects are found to be worth no more than 170l 14s : how muft it be divided among his creditors ?

Anf. O muft have 37l 15s 5d 2$\frac{5302}{10439}$q.

$$P \quad - \quad 70 \quad 15 \quad 2 \quad 2\tfrac{7498}{10439}$$
$$Q \quad - \quad 14 \quad 8 \quad 4 \quad 0\tfrac{4720}{10439}$$
$$R \quad - \quad 47 \quad 14 \quad 11 \quad 2\tfrac{3358}{10439}.$$

8. A fhip worth 900l, being entirely loft, of which $\frac{1}{8}$ belonged to S, $\frac{1}{4}$ to T, and the reft to V ; what lofs will each fuftain, fuppofing 540l of her were infured ?

Anf. S will lofe 45l, T 90l, and V 225l.

9. Four perfons, w, x, y, and z, fpent among them 25s, and agree that w fhall pay $\frac{1}{2}$ of it, x $\frac{1}{3}$, y $\frac{1}{4}$, and z $\frac{1}{5}$; that is, their fhares are to be in proportion as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$; what are their fhares ?          Anf. w muft pay 9s 8d 3$\frac{41}{77}$q.

$$x \quad - \quad 6 \quad 5 \quad 3\tfrac{53}{77}$$
$$y \quad - \quad 4 \quad 10 \quad 1\tfrac{59}{77}$$
$$z \quad - \quad 3 \quad 10 \quad 3\tfrac{1}{77}.$$

10. A detachment, confifting of 5 companies, being fent into a garrifon, in which the duty required 76 men a day ; what number of men muft be furnifhed by each company, in proportion to their ftrength; the 1ft confifting of 54 men, the 2d of 51 men, the 3d of 48 men, the 4th of 39, and the 5th of 36 men ?

Anf. The 1ft muft furnifh 18, the 2d 17, the 3d 16, the 4th 13, and the 5th 12 men*.

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* Queftions of this nature frequently occurring in military fervice, General Haviland, an officer of great merit, contrived an ingenious inftrument, for more expeditioufly refolving them ; which is diftinguifhed by the name of the inventor, being called a Haviland.

DOUBLE

# DOUBLE FELLOWSHIP.

DOUBLE FELLOWSHIP, as has been faid, is concerned in cafes in which the ftocks of partners are employed or continued for different times.

## RULE*.

MULTIPLY each perfon's ftock by the time of its continuance; then divide the quantity, as in Single Fellowfhip, into fhares in proportion to thefe products, by faying,
As the total fum of all the faid products,
Is to the whole gain or lofs, or quantity to be parted,
So is each particular product,
To the correfpondent fhare of the gain or lofs.

## EXAMPLES.

1. A had in company 50l for 4 months, and B had 60l for 5 months; at the end of which they find 24l gained: how muft it be divided between them?

$$\text{Here } \quad \underline{\begin{array}{c} 50 \\ 4 \end{array}} \quad \underline{\begin{array}{c} 60 \\ 5 \end{array}}$$
$$\underline{200} + \underline{300} = 500$$

Then, as $500 : 24 :: 200 : 9\frac{3}{5} = 9l$ 12s = A's fhare,
and as $500 : 24 :: 300 : 14\frac{2}{5} = 14 \quad 8 = $ B's fhare.

2. C and D hold a piece of ground in common, for which they are to pay 36l. C put in 23 horfes for 27 days, and D 21 horfes for 39 days; how much ought each man to pay of the rent?                       Anf. C muft pay 15l 10s 6d.
                                        D muft pay 20     9    6.

3. Three perfons, E, F, G, hold a pafture in common, for which they are to pay 30l per annum; into which E put 7 oxen for 3 months, F put 9 oxen for 5 months, and G put

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* The proof of this rule is as follows: When the times are equal, the fhares of the gain or lofs are evidently as the ftocks, as in Single Fellowfhip; and when the ftocks are equal, the fhares are as the times; therefore, when neither are equal, the fhares muft be as their products.

in 4 oxen for 12 months ; how much muſt each perſon pay
of the rent ?          Anſ. E muſt pay 5l 10s 6d $1\frac{5}{19}$q.

         F     —    11   16   10   $0\frac{8}{19}$

         G    —    12   12   7   $2\frac{6}{19}$.

4. A ſhip's company take a prize of 1000l, which they
agree to divide among them according to their pay and the
time they have been on board : now the officers and midſhip-
men have been on board 6 months, and the ſailors 3 months;
the officers have 40s a month, the midſhipmen 30s, and the
ſailors 22s a month.; moreover there are 4 officers, 12 mid-
ſhipmen, and 110 ſailors : what will each man's ſhare be ?

     Anſ. each officer muſt have 23l 2s 5d $0\frac{92}{173}$q.

      each midſhip.    —    17   6   9   $3\frac{69}{173}$ .

      each ſeaman     —    6   7   2   $0\frac{8}{173}$.

5. H, with a capital of 1000l, began trade the firſt of Ja-
nuary, and, meeting with ſuccefs in buſinefs, took in I as a
partner, with a capital of 1500l, on the firſt of March fol-
lowing. Three months after that they admit K as a third
partner, who brought into ſtock 2800l. After trading toge-
ther till the end of the year, they find there has been gained
1776l 10s : how muſt this be divided among the partners ?

       Anſ. H muſt have 457l 9s $4\frac{1}{4}$d.

        I    —     571   16   $8\frac{1}{4}$

       K    —     747   3   $11\frac{1}{4}$.

6. X, Y, and Z made a joint-ſtock for 12 months ; x at
firſt put in 20l, and 4 months after 20l more ; Y put in at
firſt 30l, at the end of 3 months he put in 20l more, and 2
months after he put in 40l more ; z put in at firſt 60l, and
5 months after he put in 10l more, 1 month after which he
took out 30l ; during the 12 months they gained 50l ; how
much of it muſt each have ?

       Anſ. x muſt have 10l 18s 6d $3\frac{40}{61}$q.

        Y    —     22   8   1   $0\frac{12}{61}$

        z    —     16   13   4   0.

SIMPLE

# SIMPLE INTEREST.

INTEREST is the premium or fum allowed for the loan, or forbearance, of money.

The money lent, or forborn, is called the Principal.

The fum of the principal and its intereft, added together, is called the Amount.

Intereft is allowed at fo much per cent. per annum ; which premium per cent. per annum, or intereft of 100l for a year, is called the rate of Intereft :—So,

When intereft is at 3 per cent. the rate is 3 ;
  —    —    —    4 per cent.    —    4 ;
  —    —    —    5 per cent.    —    5 ;
  —    —    —    6 per cent.    —    6.

But, by law, intereft ought not to be taken higher than at the rate of 5 per cent.

Intereft is of two forts ; Simple and Compound.

Simple Intereft is that which is allowed for the principal lent or forborn only, for the whole time of forbearance.

As the intereft of any fum, for any time, is directly proportional to the principal fum, and alfo to the time of continuance ; hence arifes the following general rule of calculation.

## GENERAL RULE.

As 100l is to the rate of intereft, fo is any given principal, to its intereft for one year. And again,

As 1 year is to any given time, fo is the intereft for a year, juft found, to the intereft of the given fum for that time.

OTHERWISE. Take the intereft of 1 pound for a year, which multiply by the given principal, and this product again by the time of loan or forbearance, in years and parts, for the intereft of the propofed fum for that time.

*Note,* When there are certain parts of years in the time, as quarters, or months, or days; they may be worked for, either by taking the aliquot or like parts of the intereft of a year, or by the Rule-of-Three, in the ufual way. Alfo, to divide, by 100, is done by only pointing off two figures for decimals.

## EXAMPLES.

1. To find the intereft of 230l 10s, for 1 year, at the rate of 4 per cent. per annum.

Here,

Here, As 100 : 4 :: 230l 10s : 9l 4s 4¼d.

$$\begin{array}{r} 4 \\ \hline 100\,)\,9{,}22\ \ 0 \\ 20 \\ \hline 4{\cdot}40 \\ 12 \\ \hline 4{\cdot}80 \\ 4 \\ \hline 3{\cdot}20 \end{array}$$

Anf. 9l 4s 4¼d.

2. To find the intereft of 547l 15s, for 3 years, at 5 per cent. per annum.

As 100 : 5 :: 547·75 :
Or   20 : 1 :: 547·75 : 27·3875 intereft for 1 year.

$$\begin{array}{r} 3 \\ \hline l\ \ 82{\cdot}1625 \text{ ditto for 3 years.} \\ 20 \\ \hline s\ \ 3{\cdot}2500 \\ 12 \\ \hline d\ \ 3{\cdot}00 \end{array}$$

Anf. 82l 3s 3d.

3. To find the intereft of 200 guineas, for 4 years 7 months and 25 days, at 4½ per cent. per annum.

$$\begin{array}{ll} & \qquad\quad \text{ds} \qquad\quad l \qquad\quad \text{ds} \\ 210 & \text{As } 365 : 9{\cdot}45 :: 25 : \qquad l \\ 4\tfrac{1}{2} & \text{or } \ 73 : 9{\cdot}45 :: \ 5 : \cdot 6472 \end{array}$$

$$\begin{array}{l} 840 \\ 105 \\ \hline 9{\cdot}45 \text{ intereft for 1 yr.} \\ 4 \\ \hline 37{\cdot}80 \qquad \text{ditto 4 years.} \\ 6\text{mo}=\tfrac{1}{2}\ \ 4{\cdot}725 \quad \text{ditto 6 months.} \\ 1\text{mo}=\tfrac{1}{6}\ \ \cdot 7875 \quad \text{ditto 1 month.} \\ \qquad\quad \cdot 6472 \text{ ditto 25 days.} \\ \hline l\ \ 43{\cdot}9597 \\ 20 \\ \hline s\ \ 19{\cdot}1940 \\ 12 \\ \hline d\ \ 2{\cdot}3280 \\ 4 \\ \hline q\ \ 1{\cdot}3120 \end{array}$$

$$\begin{array}{l} 5 \\ \hline 73\,)\,47{\cdot}25\,(\ \cdot 6472 \\ 345 \\ 530 \\ 19 \end{array}$$

Anf. 43l 19s 2¼d.

Ex. 4.

Ex. 4. To find the intereſt of 450l, for a year, at 5 per cent. per annum.                    Anſ. 22l 10s.

5. To find the intereſt of 715l 12s 6d, for a year, at 4½ perᵗcent. per annum.                    Anſ. 32l 4s 0¾d.

6. To find the intereſt of 720l, for 3 years, at 5 per cent. per annum.                    Anſ. 108l.

7. To find the intereſt of 355l 15s, for 4 years, at 4 per cent. per annum.                    Anſ. 56l 18s 4¾d.

8. To find the intereſt of 32l 5s 8d, for 7 years, at 4½ per cent. per annum.                    Anſ. 9l 12s 1d.

9. To find the intereſt of 170l, for 1½ year, at 5 per cent. per annum.                    Anſ. 12l 15s.

10. To find the inſurance of 205l 15s, for ¼ of a year, at 4 per cent. per annum.                    Anſ. 2l 1s 1¾d.

11. To find the intereſt of 319l 6d, for 5¾ years, at 3¾ per cent. per annum.                    Anſ. 68l 15s 9½d.

12. To find the inſurance on 107l, for 117 days, at 4¾ per cent. per annum.                    Anſ. 1l 12s 7d.

13. To find the intereſt of 17l 5s, for 117 days, at 4¾ per cent. per annum.                    Anſ. 5s 3d.

14. To find the inſurance on 712l 6s, for 8 months, at 7½ per cent. per annum.                    Anſ. 35l 12s 3½d.

*Note.* The Rules for Simple Intereſt, ſerve alſo to calculate Inſurances, or the Purchaſe of Stocks, or any thing elſe that is rated at ſo much per cent.

See alſo more on the ſubject of Intereſt, with the algebraical expreſſion and inveſtigation of the rules, at the end of the Algebra, next following.

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# COMPOUND INTEREST.

Compound Interest, called alſo Intereſt upon Intereſt, is that which ariſes from the principal and intereſt, taken together, as it becomes due, at the end of each ſtated time of payment.

Although it be not lawful to lend money at Compound Intereſt, yet in purchaſing annuities, penſions, or leaſes in reverſion, it is uſual to allow Compound Intereſt to the pur-chaſer for his ready money.

<center>RULES.</center>

1. FIND the amount of the given principal, for the time of the firſt payment, by Simple Intereſt. Then conſider this amount as a new principle for the ſecond payment, whoſe amount calculate as before. And ſo on through all the pay-ments to the laſt, always accounting the laſt amount as a new principal for the next payment. The reaſon of which is evident from the definition of Compound Intereſt. _Or elſe_,

2. Find the amount of 1 pound for the time of the firſt payment, and raiſe or involve it to the power whoſe index is denoted by the number of payments. Then that power multiplied by the given principal, will produce the whole amount. From which the ſaid principal being ſubtracted, leaves the Compound Intereſt of the ſame. As is evident from the firſt Rule.

<center>EXAMPLES.</center>

1. To find the amount of 720l, for 4 years, at 5 per cent. per annum.

Here 5 is the 20th part of 100, and the intereſt of 1l for a year is $\frac{1}{20}$ or ·05, and its amount 1·05. Therefore,

1. *By the 1ſt Rule.*

|      | l   | s  | d  |                  |
|------|-----|----|----|------------------|
| 20 ) | 720 | 0  | 0  | 1ſt yr's princip. |
|      | 36  | 0  | 0  | 1ſt yr's intereſt. |
| 20 ) | 756 | 0  | 0  | 2d yr's princip. |
|      | 37  | 16 | 0  | 2d yr's intereſt. |
| 20 ) | 793 | 16 | 0  | 3d yr's princip. |
|      | 39  | 13 | 9½ | 3d yr's intereſt. |
| 20 ) | 833 | 9  | 9½ | 4th yr's princip. |
|      | 41  | 13 | 5¾ | 4th yr's intereſt. |
| £    | 875 | 3  | 3½ | the whole amoᵗ. |

or anſ. required.

2. *By the 2d Rule.*

1·05 amount of 1l.
1·05

1·1025 2d power of it.
1·1025 ditto.

1·21550625 4th power of it.
720

875·1645
20

s 3·2900
12

d 3·4800

2. To find the amount of 50l, in 5 years, at 5 per cent. per annum, compound intereſt.     Anſ. 63l 16s 3¼d.

Ex. 3. To find the amount of 50l, in 5 years, or 10 half-years, at 5 per cent. per annum, compound intereſt, the intereſt payable half-yearly. Anſ. 64l 0s 1d.

4. To find the amount of 50l, in 5 years, or 20 quarters, at 5 per cent. per annum, compound intereſt, the intereſt payable quarterly. Anſ. 64l 2s 0¼d.

5. To find the compound intereſt of 370l, forborn for 6 years, at 4 per cent. per annum. Anſ. 98l 3s 4¼d.

6. To find the compound intereſt of 410l, forborn for 2½ years, at 4½ per cent. per annum, the intereſt payable half-yearly. Anſ. 48l 4s 11¼d.

7. To find the amount, at compound intereſt, of 217l, forborn for 2¼ years, at 5 per cent. per annum, the intereſt payable quarterly, Anſ. 242l 13s 4½d.

*Note.* See the Rules for Compound Intereſt algebraically inveſtigated, at the end of the Algebra.

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# ALLIGATION.

ALLIGATION teaches how to compound or mix together ſeveral ſimples of different qualities, ſo that the compoſition may be of ſome intermediate quality, or rate. It is commonly diſtinguiſhed into two caſes, Alligation Medial, and Alligation Alternate.

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## ALLIGATION MEDIAL.

ALLIGATION MEDIAL is the method of finding the rate or quality of the compoſition, from having the quantities and rates or qualities of the ſeveral ſimples given.

### RULE*.

MULTIPLY the quantity of each ingredient by its rate or quality; then add all the products together, and add alſo all

the

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* *Demonſtration.* The Rule is thus proved by Algebra;

Let *a*, *b*, *c* be the quantities of the ingredients,

K 2

the quantities together into another fum : then divide the former fum by the latter, that is, the fum of the products by the fum of the quantities, and the quotient will be the rate or quality of the compofition required.

<div style="text-align:center">E X A M P L E S.</div>

1. If three forts of gunpowder be mixed together, viz. 50lb at 12d a pound, 44lb at 9d, and 26lb at 8d a pound ; how much a pound is the compofition worth ?

$$\text{Here } 50, 44, 26 \text{ are the quantities,}$$
$$\text{and } 12, \quad 9, \quad 8 \text{ the rates or qualities ;}$$
$$\text{then } 50 \times 12 = 600$$
$$44 \times 9 = 396$$
$$26 \times 8 = 208$$

$$120 \;)\; 1204 \quad ( \; 10\tfrac{4}{120} = 10\tfrac{1}{30}.$$

Anf. The rate or price is $10\tfrac{1}{30}$d the pound.

2. A compofition being made of 5lb of tea at 7s per lb, 9lb at 8s 6d per lb, and $14\tfrac{1}{2}$lb at 5s 10d per lb ; what is a lb of it worth ?                          Anf. 6s $10\tfrac{1}{2}$d.

3. Mixed 4 gallons of wine at 4s 10d per gall. with 7 gallons at 5s 3d per gall. and $9\tfrac{3}{4}$ gallons at 5s 8d per gall; what is a gallon of this compofition worth?     Anf. 5s $4\tfrac{1}{4}$d.

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and $m$, $n$, $p$ their rates, or qualities, or prices;
then $am$, $bn$, $cp$ are their feveral values,
and $am + bn + cp$ the fum of their values,
alfo $a + b + c$ is the fum of the quantities,
and if $r$ denote the rate of the whole compofition,
then $\overline{a + b + c} \times r$ will be the value of the whole,
confeq. $\overline{a + b + c} \times r = am + bn + cp$,
and $r = \overline{am + bn + cp} \div \overline{a + b + c}$, which is the Rule.

*Note,* If an ounce or any other quantity of pure gold be reduced into 24 equal parts, thefe parts are called Caracts ; but gold is often mixed with fome bafe metal, which is called the Alloy, and the mixture is faid to be of fo many caracts fine, according to the proportion of pure gold contained in it ; thus, if 22 caracts of pure gold, and 2 of alloy be mixed together, it is faid to be 22 caracts fine.

If any one of the fimples be of little or no value with refpect to the reft, its rate is fuppofed to be nothing ; as water mixed with wine, and alloy with gold and filver.

<div style="text-align:right">Ex. 4.</div>

**Ex. 4.** A mealman would mix 3 bufhels of flour at 3s 5d per bufhel, 4 bufhels at 5s 6d per bufhel, and 5 bufhels at 4s 8d per bufhel : what is the worth of a bufhel of this mixture?                                                   Anf. 4s. 7½d.

5. A farmer mixes 20 bufhels of wheat at 5s the bufhel, with 36 bufhels of rye at 3s the bufhel, and 40 bufhels of barley at 2s per bufhel : how much is a bufhel of the mixture worth ?                                               Anf. 3s.

6. Having melted together 7 oz of gold of 22 caraĉts fine, 12½ oz of 21 caraĉts fine, and 17 oz of 19 caraĉts fine : I would know the finenefs of the compofition ?
                                        Anf. 20$\frac{1}{7}\frac{9}{3}$ caraĉts fine.

7. Of what finenefs is that compofition, which is made by mixing 3lb of filver of 9 oz fine, with 5lb 8oz of 10 oz fine, and 1lb 10 oz of alloy ?           Anf. 7$\frac{6}{6}\frac{1}{3}$oz fine.

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# ALLIGATION ALTERNATE.

ALLIGATION ALTERNATE is the method of finding what quantity of any number of fimples, whofe rates are given, will compofe a mixture of a given rate. So that it is the reverfe of Alligation Medial, and may be proved by it.

### RULE 1.*

1. SET the rates of the fimples in a column under each other.
2. Conneĉt, or link with a continued line, the rate of each fimple, which is lefs than that of the compound, with one, or any number, of thofe that are greater than the compound; and each greater rate with one or any number of the lefs.

3. Write

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\* *Demonft.* By conneĉting the lefs rate to the greater, and placing the difference between them and the rate alternately, the quantities refulting are fuch, that there is precifely as much gained by one quantity as is loft by the other, and therefore the gain and lofs upon the whole is equal, and is exaĉtly the propofed rate : and the fame will be true of any other two fimples managed according to the Rule.

In

3. Write the difference between the mixture rate, and that of each of the fimples, oppofite the rate with which they are linked.

4. Then if only one difference ftand againft any rate, it will be the quantity belonging to that rate; but if there be feveral, their fum will be the quantity.

### EXAMPLES.

1. A merchant would mix wines at 17s, 18s, and 22s per gallon, fo as that the mixture may be worth 20s the gallon: what quantity of each muft be taken?

$$\text{Here } 20 \begin{cases} 17 \\ 18 \\ 22 \end{cases} \quad \begin{array}{l} 2 \text{ at } 17s \\ 2 \text{ at } 18s \\ 3 + 2 = 5 \text{ at } 22s \end{array}$$

Anf. 2 gallons at 17s, 2 gallons at 18s, and 5 at 22s.

2. How much wine at 6s per gallon, and at 4s per gallon, muft be mixed together, that the compofition may be worth 5s per gallon?        Anf. 1qt, or 1gall, &c.

3. How much corn at 2s 6d, 3s 8d, 4s, and 4s 8d per bufhel, muft be mixed together, that the compound may be worth 3s 10d per bufhel?

Anf. 12 at 2s 6d, 12 at 3s 8d, 18 at 4s, and 18 at 4s 8d.

4. A goldfmith has gold of 17, 18, 22, and 24 caraɛts fine: how much muft he take of each, to make it 21 caraɛts fine?

Anf. 3 of 17, 1 of 18, 3 of 22, and 4 of 24.

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In like manner, let the number of fimples be what they will, and with how many foever every one is linked, fince it is always a lefs with a greater than the mean price, there will be an equal balance of lofs and gain between every two, and confequently an equal balance on the whole.  Q. E. D.

It is obvious, from the Rule, that queftions of this fort admit of a great variety of anfwers; for, having found one anfwer, we may find as many more as we pleafe, by only multiplying or dividing each of the quantities found, by 2, or 3, or 4, &c. the reafon of which is evident; for, if two quantities, of two fimples, make a balance of lofs and gain, with refpeɛt to the mean price, fo muft alfo the double or treble, the $\frac{1}{2}$ or $\frac{1}{3}$ part, or any other ratio of thefe quantities, and fo on *ad infinitum.*

Thefe kinds of queftions are called by algebraifts *indeterminate* or *unlimited* problems; and by an analytical procefs, theorems may be raifed that will give all the *poffible* anfwers.

Ex. 5.

Ex. 5. It is required to mix brandy at 8s, wine at 7s, cyder at 1s, and water at 0 per gallon together, fo that the mixture may be worth 5s per gallon ?

Anf. 9 gals of brandy, 9 of wine, 5 of cyder, and 5 of water.

6. How much fugar at 4d, at 6d, and at 11d per lb, muft be mixed together, fo that the compofition formed by them may be worth 7d per lb ?

Anf. 1lb, or 1 ftone, or 1 cwt, or any other equal quantity of each fort.

### RULE 2.

WHEN the whole compofition is limited to a certain quantity :

Find an anfwer as before by linking ; then fay, as the fum of the quantities, or differences thus determined, is to the given quantity ; fo is each ingredient, found by linking, to the required quantity of each.

### EXAMPLES.

1. How much gold of 15, 17, 18 and 22 caracts fine, muft be mixed together, to form a compofition of 40 oz of 20 caracts fine ?

$$
\text{Here } 20 \begin{cases} 15 \\ 17 \\ 18 \\ 22 \end{cases}
\begin{matrix}
- - - - - \ 2 \\
- - - - - \ 2 \\
- - - - - \ 2 \\
5 + 3 + 2 = 10 \\
\overline{\phantom{xx}16}
\end{matrix}
$$

Then,

as 16 : 40 :: 2 : 5
and 16 : 40 :: 10 : 25

Anf. 5 oz of 15, 17, and 18 caracts fine, and 25 oz of 22 caracts fine *.

Ex. 2.

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* A great number of queftions might be here given relating to the fpecific gravities of metals, &c. but one of the moft curious may here fuffice.

Hiero, king of Syracufe, gave orders for a crown to be made entirely of pure gold ; but fufpecting the workman had debafed it by mixing it with filver or copper, he recommended the difcovery of the fraud to the famous Archimedes, and defired to know the exact quantity of alloy in the crown.

Archimedes, in order to detect the impofition, procured two other maffes, the one of pure gold, the other of filver or copper, and each of the fame weight with the former ; and by putting each feparately

Ex. 2. A grocer has currants at 4d, 6d, 9d, and 11d per lb, and he would make a mixture of 240lb, fo that it might be afforded at 8d per lb ; how much of each fort muſt he take ?

Anſ. 72lb at 4d, 24 at 6d, 48 at 9d, and 96 at 11d.

### RULE 3*.

WHEN one of the ingredients is limited to a certain quantity ; Take the difference between each price, and the mean rate as before ; then fay,

As the difference of that ſimple, whofe quantity is given, is to the reſt of the differences feverally, fo is the quantity given, to the feveral quantities required.

### EXAMPLES.

1. How much wine at 5s, at 5s 6d, and 6s the gallon, muſt be mixed with 3 gallons at 4s per gallon, fo that the mixture may be worth 5s 4d per gallon ?

$$\text{Here } 64 \begin{cases} 48 \\ 60 \\ 66 \\ 72 \end{cases} \qquad \begin{matrix} 8 + 2 = 10 \\ 8 + 2 = 10 \\ 16 + 4 = 20 \\ 16 + 4 = 20 \end{matrix}$$

parately into a veffel full of water, the quantity of water expelled by them determined their fpecific gravities; from which, and their given weights, the exact quantities of gold and alloy in the crown may be determined.

Suppofe the weight of each crown to be 10lb, and that the water expelled by the copper or filver was 92lb, by the gold 52lb, and by the compound crown 64lb ; what will be the quantities of gold and alloy in the crown ?

The rates of the fimples are 92 and 52, and of the compound 64; therefore

$$64 \begin{vmatrix} 92 \\ 52 \end{vmatrix} \begin{matrix} 12 \text{ of copper} \\ 28 \text{ of gold} \end{matrix}$$

And the fum of thefe is 12 + 28 = 40, which fhould have been but 10; therefore by the Rule.

$$\left. \begin{matrix} 40 : 10 :: 12 : 3\text{lb of copper} \\ 40 : 10 :: 28 : 7\text{lb of gold} \end{matrix} \right\} \text{the anfwer.}$$

* In the very fame manner queſtions may be wrought when feveral of the ingredients are limited to certain quantities, by finding firſt for one limit, and then for another.

The two laſt Rules can want no demonſtration, as they evidently refult from the firſt, the reafon of which has been already explained.

Then

Then 10 : 10 :: 3 : 3 .
   10 : 20 :: 3 : 6
   10 : 20 :: 3 : 6
  Anf. 3 gallons at 5s, 6 at 5s 6d, and 6 at 6s.

2. A grocer would mix teas at 12s, 10s, and 6s per lb, with 20lb at 4s per lb : how much of each fort muſt he take to make the compofition worth 8s per lb ?
  Anf. 20lb at 4s, 10lb at 6s, 10lb at 10s, and 20lb at 12s.

3. How much gold of 15, of 17, and of 22 caraɛts fine, muſt be mixed with 5 oz of 18 caraɛts fine, fo that the compofition may be 20 caracts fine ?
  Anf. 5 oz of 15 caraɛts fine, 5 oz of 17, and 25 of 22.

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## POSITION.

Position is a method of performing certain queſtions, which cannot be refolved by the common direɛt rules. It is fometimes called Falfe Pofition, or Falfe Suppofition, becaufe it makes a fuppofition of falfe numbers, to work with the fame as if they were the true ones, and by their means difcovers the true numbers fought. It is fometimes alfo called Trial-and-Error, becaufe it proceeds by *trials* of falfe numbers, and thence finds out the true ones by a comparifon of the *errors*.

Pofition is either Single or Double.

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## SINGLE POSITION.

Single Position is that by which a queſtion is refolved by means of one fuppofition only.

Queſtions which have their refults proportional to their fuppofitions, belong to Single Pofition : fuch as thofe which require the multiplication or divifion of the number fought by any propofed number ; or when it is to be increafed or diminiſhed by itfelf, or any parts of itfelf, a certain propofed number of times.

<div align="right">RULE.</div>

### RULE.

TAKE or affume any number for that which is required, and perform the fame operations with it, as are defcribed or performed in the queftion.

Then fay, As the refult of the faid operation, is to the pofition, or number affumed; fo is the refult in the queftion, to the number fought *.

### EXAMPLES.

1. A perfon, after fpending $\frac{1}{3}$ and $\frac{1}{4}$ of his money, has yet remaining 60l; what had he at firft?

Suppofe he had at firft 120l.　　　　　Proof.

Now $\frac{1}{3}$ of 120 is 40　　　　　$\frac{1}{3}$ of 144 is　48
　　$\frac{1}{4}$ of it is　30　　　　　　$\frac{1}{4}$ of 144 is　36

their fum is 70　　　　　　their fum　84
which taken from 120　　　　　taken from　144

leaves 50　　　　　　leaves 60 as
　　　　　　　　　　　　per queftion.

Then, 50 : 120 :: 60 : 144, the Anfwer.

2. What number is that, which multiplied by 7, and the product divided by 6, the quotient may be 14?　　Anf. 12.

3. What number is that, which being increafed by $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$, of itfelf, the fum fhall be 125?　　　　　Anf. 60.

4. A general, after fending out a foraging $\frac{1}{2}$ and $\frac{1}{3}$ of his men, had yet remaining 700: what number had he in command?　　　　　　　　　　　　Anf. 4200.

5. A gentleman diftributed 78 pence among a number of poor people, confifting of men, women, and children; to each man he gave 6d, to each woman 4d, and to each child 2d: moreover there were twice as many women as men, and thrice as many children as women. How many were there of each?　　Anf. 3 men, 6 women, and 18 children.

---

* The reafon of this Rule is evident, becaufe it is fuppofed that the refults are proportional to the fuppofitions.

Thus, $nx : x :: na : a$,

or $\dfrac{x}{n} : x :: \dfrac{a}{n} : a$,

or $\dfrac{x}{n} + \dfrac{x}{m}$, &c $: x :: \dfrac{a}{n} + \dfrac{a}{m}$, &c $: a$,

and fo on.

Ex. 6.

Ex. 6. One being afked his age, faid, if $\frac{3}{5}$ of the years I have lived, be multiplied by 7, and $\frac{4}{9}$ of them be added to the product, the fum will be 292. What was his age ?

Anf. 60 years.

---

## DOUBLE POSITION.

Double Position is the method of refolving certain queftions by means of two fuppofitions of falfe numbers.

To the Double Rule of Pofition belong fuch queftions as have their refults not proportional to their pofitions : fuch are thofe, in which the numbers fought, or their parts or their multiples, are increafed or diminifhed by fome given abfolute number, which is no known part of the number fought.

### RULE*.

Take or affume any two convenient numbers, and proceed with each of them feparately, according to the conditions of the queftion, as in Single Pofition ; and find how much each refult is different from the refult mentioned in the queftion, calling thefe differences the *errors*, noting alfo whether the refults are too great or too little.

Then

---

* *Demonftr.* The Rule is founded on this fuppofition, namely, that the firft error is to the fecond, as the difference between the true and firft fuppofed number, is to the difference between the true and fecond fuppofed number ; when that is not the cafe, the exact anfwer to the queftion cannot be found by this Rule.—That the Rule is true, according to that fuppofition. may be thus proved.

Let $a$ and $b$ be the two fuppofitions, and A and B their refults, produced by fimilar operations ; alfo $r$ and $s$ their errors, or the differences between the refults A and B from the true refult N ; and let $x$ denote the number fought, anfwering to the true refult N of the queftion.

Then is $N - A = r$, and $N - B = s$. And, according to the fuppofition on which the Rule is founded. $r : s :: x - a : x - b$ ; hence, by multiplying extremes and means, $rx - rb = sx - sa$ ; then, by tranfpofition, $rx - sx = rb - sa$ ; and, by divifion,

$$x = \frac{rb - sa}{r - s} = \text{the number fought, which is the Rule when the}$$

refults are both too little.

If

Then multiply each of the faid errors by the contrary fuppofition, namely, the firft pofition by the fecond error, and the fecond pofition by the firft error. Then,

If the errors are alike, divide the difference of the products by the difference of the errors, and the quotient will be the anfwer.

But if the errors are unlike, divide the fum of the products by the fum of the errors, for the anfwer.

*Note,* The errors are faid to be alike, when they are either both too great or both too little ; and unlike, when one is too great and the other too little.

### EXAMPLES.

1. What number is that, which being multiplied by 6, the product increafed by 18, and the fum divided by 9, the quotient fhall be 20 ?

Suppofe the two numbers 18 and 30. Then,

| First Pofition. | | Second Pofition. | | Proof, |
|---|---|---|---|---|
| 18 | Suppofe | 30 | | 27 |
| 6 | mult. | 6 | | 6 |
| 108 | | 180 | | 162 |
| 18 | add | 18 | | 18 |
| 9 ) 126 | | 9 ) 198 | | 9 ) 180 |
| 14 | refults | 22 | | 20 |
| 20 | true ref. | 20 | | |
| + 6 | errors unlike | — 2 | | |

2d pof. 30 mult.    18 1ft pof.

Er- ⌉ 2 180
rors ⌡ 6   36      36

fum 8 ) 216   fum of products

     27   Anfwer fought.

### RULE 2.

FIND, by trial, two numbers, as near the true number as convenient, and operate with them as in the queftion; marking the errors which arife from each of them.

Mul-

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If the refults be both too great, fo that A and B are both greater than N ; then N — A = — $r$, and N — B = — $s$, or $r$ and $s$ are both negative; hence — $r$ : — $s$ :: $x$ — $a$ : $x$ — $b$, but — $r$ : — $s$ :: + $r$ : + $s$, therefore $r$ : $s$ :: $x$ — $a$ : $x$ — $b$, and the reft will be exactly as in the former cafe.

But

Multiply the difference of the two numbers affumed, or found by trial, by the leaft error, and divide the product by the difference of the errors, when they are alike, but by their fum when they are unlike.

Add the quotient, laft found, to the number belonging to the leaft error, when that number is too little, but fubtract it when too great, and the refult will give the true quantity fought*.

<center>EXAMPLES.</center>

1. So, the foregoing example worked by this 2d rule, will be as follows :

<pre>
    30 pofitions 18 ;        their dif.  12
  — 2 errors + 6 ;          leaft error  2
              fum of errors 8 ) 24 ( 3 fubtr.
                      from the pofition  30
              leaves the anfwer  27
</pre>

2. A fon afking his father how old he was, received this Anfwer : Your age is now one-fourth of mine; but 5 years ago, your age was only one-fifth of mine. What then are their two ages ?                              Anf. 20 and 80.

3. A workman was hired for 30 days, at 2s 6d per day, for every day he worked; but with this condition, that for every day he played, he fhould forfeit 1s. Now it fo happened, that upon the whole he had 2l 14s to receive. How many of the days did he work ?                    Anf. 24.

4. A and B began to play together with equal fums of money : A firft won 20 guineas, but afterwards loft back $\frac{2}{3}$ of what he then had; after which, B had 4 times as much as A. What fum did each begin with ?          Anf. 100 guineas.

5. Two perfons, A and B, have both the fame income. A faves $\frac{1}{5}$ of his; but B, by fpending 50l per annum more than A, at the end of 4 years finds himfelf 100l in debt. What doth each receive and fpend per annum ?

Anf. They receive 125l per annum; alfo A fpends 100l, and B fpends 150l per annum.

---

But if one refult A only be too little, and the other B too great, or one error r pofitive, and the other s negative, then the theorem becomes $x = \dfrac{rb + sa}{r + s}$, which is the Rule in this cafe, or when the errors are unlike.

* For fince, by the fuppofition, $r : s :: x - a : x - b$, therefore by divifion, $r - s : s :: b - a : x - b$, which is the 2d Rule.

## PERMUTATIONS AND COMBINATIONS.

PERMUTATION is the altering, changing, or varying the position or order of things ; or the shewing how many different ways they may be placed.—This is otherwise called Alternation, Changes, or Variation; and the only thing to be regarded here, is the order they stand in ; for no two parcels are to have all their quantities placed in the same situation: as, how many changes may be rung on a number of bells, or how many different ways any number of persons may be placed, or how many several variations may be made of any number of letters, or any other things proposed to be varied.

COMBINATION is the shewing how often a less number of things can be taken out of a greater, and combined together, without considering their places, or the order they stand in. This is sometimes called Election or Choice ; and here every parcel must be different from all the rest, and no two are to have precisely the same quantities or things.

*Combinations of the same Form*, are those in which there are the same number of quantities, and the same repetitions: thus, *aabc*, *bbcd*, *ccde*, are of the same form ; *aabc*, *abbb*, *aabb*, are of different forms.

*Composition of Quantities*, is the taking a given number of quantities, out of as many equal rows of different quantities, one out of every row, and combining them together.

Some illustrations of these definitions are in the following Problems :

### PROBLEM I.

*To assign the Number of Permutations, or Changes, that can be made of any Given Number of Things, all different from each other.*

### RULE*.

MULTIPLY all the terms of the natural series of numbers, from 1 up to the given number, continually together, and the last product will be the answer required.

. EXAM-

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* The reason of the Rule may be shewn thus : any one thing *a* is capable only of one position, as *a*.

Any two things *a* and *b*, are only capable of two variations ; as *ab*, *ba* ; whose number is expressed by 1 × 2.

If

EXAMPLES.

1. How many changes may be rung on 6 bells?

$$\begin{array}{r} 1 \\ \underline{2} \\ 2 \\ \underline{3} \\ 6 \\ \underline{4} \\ 24 \\ \underline{5} \\ 120 \\ \underline{6} \\ 720 \end{array} \text{ the Anfwer.}$$

Or $1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720$ the Anfwer.

2. How many days can 7 perfons be placed in a different pofition at dinner?                    Anf. 5040 days.

3. How many changes may be rung on 12 bells, and what time would it require, fuppofing 10 changes to be rung in 1 minute, and the year to confift of 365 days, 5 hours, and 49 minutes?

Anf. 479001600 changes, and 91 years, 26 days, 22 hours, 41 minutes.

4. How many changes may be made of the words in the following verfe: *Tot tibi funt dotes, virgo, quot fidera cœlo?*
Anf. 40320 changes.

---

If there be three things, *a, b,* and *c*; then any two of them, leaving out the 3d, will have $1 \times 2$ variations; and confequently, when the 3d is taken in, there will be $1 \times 2 \times 3$ variations.

In the fame manner, when there are 4 things, every three, leaving out the 4th, will have $1 \times 2 \times 3$ variations; confequently by taking in fuccefively the 4 left out, there will be $1 \times 2 \times 3 \times 4$ variations. And fo on as far as we pleafe.

## PROBLEM II.

*Any Number of different Things being given; to find how many Changes can be made out of them, by taking a Given Number of Quantities at a Time.*

### RULE*.

TAKE a feries of numbers, beginning at the number of things given, and decreafing by 1 to the number of quantities to be taken at a time, and the product of all the terms will be the anfwer required.

### EXAMPLES.

1. How many changes may be rung with 3 bells out of 8 ?

$$
\begin{array}{r}
8 \\
7 \\
\hline
56 \\
6 \\
\hline
336 \text{ the Anfwer.}
\end{array}
$$

Or, $8 \times 7 \times 6$ ( $= 3$ terms) $= 336$ the Anfwer.

2. How many words can be made with 5 letters of the alphabet, admitting that a number of confonants alone will not make a word ?                     Anf. 5100480.

PRO-

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* This Rule, expreffed in algebraic terms, is as follows :

$m \times \overline{m - 1} \times \overline{m - 2} \times \overline{m - 3}$ &c, to $n$ terms: where $m =$ the number of things given, and $n =$ the quantities to be taken at a time.

In order to demonftrate the Rule, it will be proper to premife the following Lemma :

LEMMA. The number of changes of $m$ things, taken $n$ at a time, is equal to $m$ changes of $m - 1$ things, taken $n - 1$ at a time.

*Demonftr.* Let any five quantities, $a\ b\ c\ d\ e$ be given.

Firft, leave out the $a$, and let $v =$ the number of all the variations of every two, $bc$, $bd$, &c, that can be taken out of the four remaining quantities $b\ c\ d\ e$.

Now, let $a$ be put in the firft place of each of them; $a$, $b$, $c$, $a$, $b$, $d$, &c, and the number of changes will ftill remain the fame ; that is, $v =$ the number of variations of every 3 out of the 5, $a\ b\ c\ d\ e$, when $a$ is firft.

In

## PROBLEM III.

*Any Number of Things being given; of which there are several given Things of one Sort, and several of another, &c; To find how many Changes can be made out of them all.*

### RULE*.

TAKE the feries 1 × 2 × 3 × 4, &c, up to the number of things given, and find the product of all the terms.

Take the feries 1 × 2 × 3 × 4, &c, up to the number of given things of the firft fort, and the feries 1 × 2 × 3 × 4, &c, up to the number of given things of the fecond fort, &c.

Divide

---

In like manner. if *b*, *c*, *d*, *e* be fucceffively left out, the number of variations of all the two's will alfo be = *v*: and putting *b*, *c*, *d*, *e* refpectively in the firft place, to make 3 quantities out of 5, there will ftill be *v* variations, as before.

But thefe are all the variations that can happen of 3 things out of 5, when *a*, *b*, *c*. *d*, *e* are fucceffively put firft; and therefore the fum of all thefe is the fum of all the changes of 3 things out of 5.

But the fum of thefe is fo many times *v*, as is the number of things; that is 5*v*, or *mv*, = all the changes of 3 times out of 5.

And the fame way of reafoning may be applied to any numbers whatever.

*Demon. of the Rule.* Let any 7 things *a b c d e f g* be given, and let 3 be the number of quantities to be taken.

Then *m* = 7. and *n* = 3.

Now, it is evident, that the number of changes that can be made by taking 1 by 1 out of 5 things, will be 5, which let = *v*.

Then, by the Lemma, when *m* = 6. and *n* = 2, the number of changes will be = *mv* = 6 × 5; which let be = *v* a fecond time.

Again, by the Lemma, when *m* = 7 and *n* = 3, the number of changes is *mv* = 7 × 6 × 5; that is *mv* = *m* × (*m* — 1) × (*m* — 2), continued to 3, or *n* terms.

And the fame may be fhewn for any other numbers.

* This Rule is expreffed in terms thus :

$$\frac{1 \times 2 \times 3 \times 4 \times 5, \&c, \text{ to } m}{1 \times 2 \times 3, \&c \text{ to } p \times 1 \times 2 \times 3, \&c \text{ to } q, \&c.}$$

where *m* = the number of things given. *p* = the number of things of the firft fort, *q* = the number of things of the fecond fort, &c.

*The Demonftration* may be fhewn as follows :

Any 2 quantities, *a b*, both different, admit of 2 changes : but if the quantities are the fame, or *a b* becomes *a a*, there will be only one pofition; which may be expreffed by $\frac{1 \times 2}{1 \times 2} = 1$.

Divide the product of all the terms of the firſt ſeries by the joint product of all the terms of the remaining ones, and the quotient will be the anſwer required.

EXAMPLES.

1. How many variations can be made of the letters in the word Bacchanalia?

$$1 \times 2 \; (= \text{number of } c\text{'s}) = 2$$
$$1 \times 2 \times 3 \times 4 \; (= \text{number of } a\text{'s}) = 24$$
$$1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \times 11$$
$$(= \text{number of letters in the word}) = 39916800$$

$$2 \times 24 = 48 \;) \; 39916800 \;(\; 831600 \text{ the Anſwer.}$$
$$\begin{array}{l} 151 \\ 76 \\ 288 \\ \hline \end{array}$$

---

Any 3 quantities, *a b c*, all different from each other, afford 6 variations; but if the quantities be all alike, or *a b c* becomes *a a a*, then the 6 variations will be reduced to 1; which may be expreſſed by $\dfrac{1 \times 2 \times 3}{1 \times 2 \times 3} = 1$. Again, if two of the quantities only are alike, or *a b c* becomes *a a c*; then the 6 variations will be reduced to theſe 3, *a a c, c a a,* and *a c a*; which may be expreſſed by $\dfrac{1 \times 2 \times 3}{1 \times 2} = 3$.

Any 4 quantities, *a b c d*, all different from each other, will admit of 24 variations. But if the quantities be the ſame, or *a b c d* becomes *a a a a*, the number of variations will be reduced to one; which is $= \dfrac{1 \times 2 \times 3 \times 4}{1 \times 2 \times 3 \times 4} = 1$.

Again, if three of the quantities only be the ſame, or *a b c d* becomes *a a a b*, the number of variations will be reduced to theſe 4, *a a a b, a a b a, a b a a,* and *b a a a*; which is $= \dfrac{1 \times 2 \times 3 \times 4}{1 \times 2 \times 3} = 4$.

And thus it may be ſhewn, that if two of the quantities be alike, or the 4 quantities be *a a b c*, the number of variations will be reduced to 12; which may be expreſſed by $\dfrac{1 \times 2 \times 3 \times 4}{1 \times 2} = 12$.

And by reaſoning in the ſame manner, it will appear, that the number of changes which can be made of the quantities *a b b c c*, is equal to 60; which may be expreſſed by $\dfrac{1 \times 2 \times 3 \times 4 \times 5 \times 6}{1 \times 2 \times 1 \times 2 \times 3} = 60$. And ſo on for any other quantities whatever.

2. How

2. How many different numbers can be made of the following figures, 1220005555 ?                    Anf. 12600.

3. How many varieties will take place in the fucceffion of the following mufical notes, fa, fa, fa, fol, fol, la, mi, fa ?

Anf. 3360.

## PROBLEM IV.

*To find the Changes of any Given Number of Things, taking a Given Number at a Time; in which there are feveral Given Things of one Sort, feveral of another, &c.*

### RULE*.

FIND all the different forms of combination of all the given things, taken as many at a time as in the queftion.

Find the number of changes in any form, and multiply it by the number of combinations in that form.

Do the fame for every diftinct form, and the fum of all the products will give the whole number of changes required.

### EXAMPLES.

1. How many alternations, or changes, can be made of every four letters out of thefe 8, *aaabbbcc* ?

| No. of forms. | No. of changes. |
|---|---|
| $a^3b$, $a^3c$, $b^3a$, $b^3c$ . . . . . . . . | 4 |
| $a^2b^2$, $a^2c^2$, $b^2c^2$ . . . . . . . . | 6 |
| $a^2bc$, $b^2ac$, $c^2ab$ . . . . . . . . | 12 |

---

* The reafon of this Rule is plain from what has been fhewn before, and the nature of the problem.

*A Rule for finding the Number of Forms.*

1. PLACE the things fo, that the greateft indices may be firft, and the reft in order.

2. Begin with the firft letter, and join it to the fecond, third, fourth, &c. to the laft.

3. Then take the fecond letter, and join it to the third, fourth, &c. to the laft. And fo on, till they are entirely exhaufted, always remembering to reject fuch combinations as have occurred before; and this will give the combinations of all the two's.

4. Join the firft letter to every one of the two's, and the fecond, third, &c. as before; and it will give the combinations of all the three's.

5. Proceed in the fame manner to get the combinations of all the four's, &c, and you will at laft get all the feveral forms of combination, and the number in each form.

L 2                                    Therefore

$$\text{Therefore}\begin{cases} 4 \times 4 = 16 \\ 3 \times 6 = 18 \\ 3 \times 12 = \underline{36} \end{cases}$$

$$\overline{70} = \text{number of changes required.}$$

2. How many changes can be made of every 8 letters out of these 10 ; *aaaabbccde* ? Anf. 22260.

3. How many different numbers can be made out of 1 unit, 2 two's, 3 three's, 4 four's, and 5 five's ; taken 5 at a time ? Anf. 2111.

### PROBLEM V.

*To find the Number of Combinations of any Given Number of Things, all different from each other, taken any Given Number at a Time.*

### RULE.*

TAKE the series 1, 2, 3, 4, &c, up to the number to be taken at a time, and find the product of all the terms.

Take

---

* This Rule, expreffed algebraically, is;

$$\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \, \&c, \text{ to } n \text{ terms ; where } m \text{ is the}$$

number of given quantities, and $n$ thofe to be taken at a time.

*Demonftr. of the Rule.* 1. Let the number of things to be taken at a time be 2, and the things to be combined $= m$.

Now, when $m$, or the number of things to be combined, is only two, as $a$ and $b$, it is evident that there can be but one combination, as $ab$ ; but if $m$ be increafed by one; or the letters to be combined be 3, as $a$, $b$, $c$; then it is plain that the number of combinations will be increafed by 2, fince with each of the former letters $a$ and $b$, the new letter $c$ may be joined. In this cafe therefore, it is evident that the whole number of combinations will be truly expreffed by $1 + 2$.

Again, if $m$ be increafed by one letter more, or the whole number of letters be four, as $a$, $b$, $c$, $d$; then it will appear that the whole number of combinations muft be increafed by 3, fince with each of the preceding letters the new letter $d$ may be combined. The combinations, therefore, in this cafe, will be truly expreffed by $1 + 2 + 3$. And in the fame manner it may be fhewn that the whole number of combinations of 2, in 5 things, will be $1 + 2 + 3 + 4$; of 2, in 6 things, $1 + 2 + 3 + 4 + 5$: and of 2, in 7 things, $1 + 2 + 3 + 4 + 5 + 6$, &c. : whence, univerfally, the number of combinations

of

Take a feries of as many terms, decreafing by 1, from the given number, out of which the election is to be made, and find the product of all the terms.

Divide the laft product by the former, and the quotient will be the number fought.

## EXAMPLES.

1. How many combinations can be made of 6 letters out of ten?

$1 \times 2 \times 3 \times 4 \times 5 \times 6$ ( = the number to be taken at a time ) = 720.

$10 \times 9 \times 8 \times 7 \times 6 \times 5$ ( = fame number from 10) = 151200.

---

of $m$ things, taken 2 by 2, is $= 1 + 2 + 3 + 4 + 5 + 6$, &c. to $(m - 1)$ terms.

But the fum of this feries is $= \dfrac{m}{1} \times \dfrac{m-1}{2}$; which is the famé as the rule.

2. Let now the number of quantities in each combination be fuppofed to be three.

Then it is plain, that when $m = 3$, or the things to be combined are $a$, $b$, $c$, there can be only one combination. But if $m$ be increafed by 1, or the things to be combined are 4, as $a$, $b$, $c$, $d$, then will the number of combinations be increafed by 3 : fince 3 is the number of combinations of 2 in all the preceding letters, $a$, $b$, $c$, and with each two of thefe the new letter $d$ may be combined.

The number of combinations, therefore, in this cafe, is $1 + 3$.

Again, if $m$ be increafed by one more. or the number of letters be fuppofed 5 ; then the former number of combinations will be increafed by 6, that is, by all the combinations of 2 in the 4 preceding letters, $a$, $b$, $c$, $d$; fince, as before, with each two of thefe the new letter $e$ may be combined.

The number of combinations, therefore, in this cafe, is $1 + 3 + 6$.

Whence, univerfally, the number of combinations of $m$ things, taken 3 by 3, is $1 + 3 + 6 + 10$, &c. to $m - 2$ terms.

But the fum of this feries is $= \dfrac{m}{1} \times \dfrac{m-1}{2} \times \dfrac{m-2}{3}$; which is the fame as the rule.

And the fame thing will hold, let the number of things to be taken at a time be what it will ; therefore the number of combinations of $m$ things, taken $n$ at a time, will be $=$

$\dfrac{m}{1} \times \dfrac{m-1}{2} \times \dfrac{m-2}{3} \times \dfrac{m-3}{4}$, &c, to $n$ terms.    Q. E. D.

L 3

Then

Then 720 ) 151200 ( 210 the Anfwer.
           1440
           ——
            720
            720

2. How many combinations can be made of 2 letters out of the 24 letters of the alphabet ?                Anf. 276.

3. A general, who had often been fuccefsful in war, was afked by his king what reward he fhould confer upon him for his fervices ; the general only defired a farthing for every file, of 10 men in a file, which he could make with a body of 100 men ; what is the amount in pounds fterling ?

Anf. 18031572350l 9s 2d.

## PROBLEM VI.

*To find the Number of Combinations of any Given Number of Things, by taking any Given Number at a Time ; in which there are feveral Things of one Sort, feveral of another, &c.*

### RULE.

FIND, by trial, the number of different forms which the things to be taken at a time will admit of, and the number of combinations there are in each.

Add all the combinations, thus found, together, and the fum will be the number required.

### EXAMPLES.

1. Let the things propofed be *a a a b b c* ; it is required to find the number of combinations made of every 3 of thefe quantities ?

Forms.                                        Combinations.

$a^3$ . . . . . . . . . . . . . 1

$a^2b$, $a^2c$, $b^2a$, $b^2c$ . . . . . . . . 4

$a b c$ . . . . . . . . . . . . . 1

Number of combinations required = 6

2. Let *a a a b b b c c* be propofed ; it is required to find the number of combinations of thefe quantities, taken 4 at a time ?                Anf. 10.

3. How many combinations are there in *a a a a b b c c d e*, taking 8 at a time ?                Anf. 13.

4. How

4. How many combinations are there in *a a a a a b b b b b cccc d dd d e e e e f f f g*, taking 10 at a time ?       Anf. 2819.

### PROBLEM VII.

*To find the Compofitions of any Number, in an equal Number of Sets, the Things themfelves being all different.*

### RULE*.

MULTIPLY the number of things in every fet continually together, and the product will be the anfwer required.

### EXAMPLES.

1. Suppofe there are four companies, in each of which there are 9 men ; it is required to find how many ways 9 men may be chofen, one out of each company ?

$$
\begin{array}{r}
9 \\
9 \\
\hline
81 \\
9 \\
\hline
729 \\
9 \\
\hline
6561 \text{ the Anfwer.}
\end{array}
$$

Or, $9 \times 9 \times 9 \times 9 = 6561$ the Anfwer.

Ex. 2.

---

* *Demonftr.* Suppofe there are only two fets; then, it is plain, that every quantity of the one fet being combined with every quantity of the other, will make all the compofitions, of two things in thefe two fets ; and the number of thefe compofitions is evidently the product of the number of quantities in one fet by that in the other.

Again, fuppofe there are three fets ; then the compofition of two, in any two of the fets, being combined with every quantity of the third, will make all the compofitions of three in the three fets. That is, the compofitions of two, in any two of the fets, being multiplied by the number of quantities in the remaining fet, will produce the compofitions of three in the three fets ; which is evidently the continual product of all the three numbers in the three fets.

And the fame manner of reafoning will hold, let the number of fets be what it will. . Q. E. D.

The

Ex. 2. Suppofe there are 4 companies; in one of which there are 6 men, in another 8, and in each of the other two 9; what are the choices, by a compofition of 4 men, one out of each company? Anf. 3888.

3. How many changes are there in throwing 5 dice? Anf. 7776.

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# PRACTICAL QUESTIONS in ARITHMETIC.

Quest. 1. The fwifteft velocity of a cannon-ball, is about 2000 feet in a fecond of time. Then in what time, at that rate, would fuch a ball be in moving from the earth to the fun; admitting the diftance to be 100 millions of miles, and the year to contain 365 days 6 hours?
Anf. $8\frac{4808}{13149}$ years.

Quest. 2. What is the ratio of the velocity of light to that of a cannon-ball, which iffues from the gun with a velocity of 1500 feet per fecond; light paffing from the fun to the earth in $7\frac{1}{2}$ minutes? Anf. the ratio of $782222\frac{2}{9}$ to 1.

Quest. 3. The flow or parade-ftep being 70 paces per minute, at 28 inches each pace, it is required to determine at what rate per hour that movement is? Anf. $1\frac{11}{32}$ miles.

Quest. 4. The quick-time or ftep, in marching, being 2 paces per fecond, or 120 per minute, at 28 inches each; then at what rate per hour does a troop march on a route, and how long will they be in arriving at a garrifon 20 miles diftant, allowing a halt of one hour by the way to refrefh?
Anf. $\begin{cases} \text{the rate is } 3\frac{2}{11} \text{ miles an hour,} \\ \text{and the time } 7\frac{2}{7} \text{ hr, or 7 h. } 17\frac{1}{7} \text{ min.} \end{cases}$

Quest. 5. Two perfons, A and B, being on oppofite fides of a wood, which is 268 yards about, they begin to go round it, both the fame way, at the fame inftant of time; A

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The doctrine of permutations, combinations, &c, is of very extenfive ufe in different parts of the Mathematics: particularly in the calculation of annuities and chances. The fubject might have been purfued to a much greater length; but what is here done will bef ound fufficient for moft of the purpofes to which things of this nature are applicable.

goes

goes at the rate of 11 yards per minute, and B 34 yards in 3 minutes ; the queſtion is, how many times will the wood be gone round before the quicker overtake the ſlower ?

<div align="right">Anſ. 17 times.</div>

QUEST. 6. To determine how far 500 millions of guineas will reach, when laid down in a ſtraight line touching one another ; ſuppoſing each guinea to be an inch in diameter, as it is very nearly. Anſ. 7891 miles, 728 yds, 2 ft 8 in.

QUEST 7. A wall was to be built 700 yards long in 29 days. Now, after 12 men had been employed on it for 11 days, it was found that they had completed only 220 yards of the wall It is required then to determine how many men muſt be added to the former, that the whole number of them may juſt finiſh the wall in the time propoſed, at the ſame rate of working. Anſ. 4 men to be added.

QUEST. 8. The hour and minute hand of a clock are exactly together at 12 o'clock ; when are they next together ?

<div align="right">Anſ. At $1\frac{1}{11}$ hr, or 1 hr $5\frac{5}{11}$ min.</div>

QUEST. 9. A perſon after ſpending 10l more than $\frac{1}{3}$ of his yearly income, had then remaining 15l more than the half of it ; what was his income ? Anſ. 150l.

QUEST. 10. A perſon who was poſſeſſed of a $\frac{3}{5}$ ſhare of a copper mine, ſold $\frac{3}{4}$ of his intereſt in it for 1710l : what was the reputed value of the whole at the ſame rate ?

<div align="right">Anſ. 3800l.</div>

QUEST. 11. A can do a piece of work alone in 10 days, and B alone in 13 ; in what time will they both together perform a like quantity of work ? Anſ. $5\frac{15}{23}$ days.

QUEST. 12. A perſon bought 120 oranges at 2 a penny, and 120 more at 3 a penny ; after which, ſelling them out again at 5 for 2 pence, whether did he gain or loſe by the bargain ? Anſ. he loſt 4 pence.

QUEST. 13. If a gentleman whoſe annual income is 1200l, ſpend 20 guineas a week ; whether will he ſave or run in debt, and how much in the year ? Anſ. ſave 108l.

QUEST. 14. In the latitude of London, the diſtance round the earth, meaſured on the parallel of latitude, is about 15550 miles; now as the earth turns round in 23 hours 56 minutes, at what rate per hour is the city of London carried by this motion from weſt to eaſt ?

<div align="right">Anſ. $649\frac{259}{359}$ miles an hour.</div>

QUEST. 15. If a quantity of proviſions ſerves 1500 men 12 weeks, at the rate of 20 ounces a day for each man; how

<div align="right">many</div>

many men will the fame provifions maintain for 20 weeks, at the rate of 8 ounces a day for each man? Anf. 2250 men.

Quest. 16. If 1000 men befieged in a town, with provifions for 5 weeks, allowing each man 16 ounces a day, be reinforced with 500 men more; and fuppofing that they cannot be relieved till the end of 8 weeks, how many ounces a day muft each man have, that the provifion may laft that time? Anf. $6\frac{2}{3}$ ounces.

Quest. 17. A father left his fon a fortune, $\frac{1}{4}$ of which he ran through in 8 months: $\frac{3}{7}$ of the remainder lafted him 12 months longer; after which he had bare 410l left. What fum did the father bequeath the fon? ·

Anf. 956l 13s 4d.

Quest. 18. A perfon, looking on his watch, was afked what was the time of the day, who anfwered, It is between 4 and 5; but a more particular anfwer being required, he faid that the hour and minute hands were then exactly together: What was the time? Anf. $21\frac{9}{11}$ min. paft 4.

Quest. 19. If 20 men can perform a piece of work in 12 days, how many men will accomplifh another thrice as large in one-fifth of the time? Anf. 300.

Quest. 20. A younger brother received 6300l, which was juft $\frac{7}{9}$ of his elder brother's fortune: What was the father worth at his death? Anf. 14400l.

Quest. 21. A perfon, making his will, gave to one child $\frac{13}{20}$ of his eftate, and the reft to another. When thefe legacies came to be paid, the one turned out 600l more than the other: What did the teftator die worth? Anf. 2000l.

Quest. 22. A father devifed $\frac{7}{18}$ of his eftate to one of his fons, and $\frac{7}{18}$ of the refidue to another, and the furplus to his relict for life. The children's legacies were found to be 257l 3s 4d different: Pray what money did he leave the widow the ufe of? Anf. 635l 0s $10\frac{3}{8}$d.

Quest. 23. Two perfons, A and B, travel between London and Exeter. A leaves Exeter at 8 o'clock in the morning, and walks at the rate of 3 miles an hour, without intermiffion; and B fets out from London at 4 o'clock the fame evening, and walks for Exeter at the rate of 4 miles an hour conftantly. Now, fuppofing the diftance between the two cities to be 130 miles, whereabouts on the road will they meet? Anf. $69\frac{3}{7}$ miles from Exeter.

Quest.

QUEST. 24. Two perfons, A and B, travel between London and Lincoln, diftant 100 miles, A from London, and B from Lincoln, at the fame inftant. After 7 hours they meet on the road, when it appeared that A had rode 1½ miles an hour more than B. At what rate per hour then did each of the travellers ride ?

Anf. A. 7$\frac{25}{28}$, and B 6$\frac{11}{28}$ miles.

QUEST. 25. The clocks of Italy go on to 24 hours : Then how many ftrokes do they ftrike in one complete revolution of the index ?  Anf. 300.

QUEST. 26. One hundred eggs being placed on the ground, in a ftraight line, at the diftance of a yard from each other : How far will a perfon travel who fhall bring them one by one to a bafket, which is placed at one yard from the firft egg ?  Anf. 10100 yds, or 5 miles and 1300 yds.

QUEST. 27. One Seffa, an Indian, having invented the game of chefs, fhewed it to his prince, who was fo delighted with it, that he promifed him any reward he fhould afk ; on which Seffa requefted that he might be allowed one grain of wheat for the firft fquare on the chefs board, 2 for the fecond, 4 for the third, and fo on, doubling continually, to 64, the whole number of fquares. Now, fuppofing a pint to contain 7680 of thefe grains, and one quarter or 8 bufhels to be worth 27s 6d, it is required to compute the value of all the corn ?  Anf. 6450468216285l 17s 3d 3$\frac{3273757}{3276 8}$q.

QUEST. 28. Two young gentlemen, without private fortune, obtain commiffions at the fame time, and at the age of 18. One thoughtlefsly fpends 10l a year more than his pay ; but, fhocked at the idea of not paying his debts, gives his creditor a bond for the money, at the end of every year, and alfo infures his life for the amount ; each bond cofts him 30 fhillings, befides the lawful intereft of 5 per cent, and to infure his life cofts him 6 per cent.

The other, having a proper pride, is determined never to run in debt ; and, that he may affift a friend in need, perfeveres in faving 10l every year, for which he obtains an intereft of 5 per cent, which intereft is every year added to his favings, and laid out, fo as to anfwer the effect of compound intereft.

Suppofe thefe two officers to meet at the age of 50, when each receives from Government 400l. per annum ; that the one, feeing his paft errors, is refolved in future to fpend no

more

more than he actually has, after paying the interest for what he owes, and the infurance on his life.

The other, having now fomething before hand, means in future to fpend his full income, without increafing his ftock.

It is defirable to know how mucn each has to fpend per annum, and what money the latter has by him to affift the diftreffed, or leave to thofe who deferve it ?

# END of ARITHMETIC,

## OR OF

## PART I. VOL. I.

# LOGARITHMS

### AND

# ALGEBRA:

#### BEING

#### PART II. OF VOL. I.

# CONTENTS of PART II.

## VOL. I.

### LOGARITHMS.

### ALGEBRA.

Geo-

# CONTENTS.

# LOGARITHMS

AND

# ALGEBRA.

## OF LOGARITHMS*.

LOGARITHMS are numbers fo contrived, and adapted to other numbers, that the fums and differences of the former fhall correfpond to, and fhew, the products and quotients of the latter.

Or, more generally, logarithms are the numerical exponents of ratios; or a feries of numbers in arithmetical progreffion, anfwering to another feries of numbers in geometrical progreffion.

Thus

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* The invention of Logarithms is the undoubted right of Lord Napier, Baron of Merchifton, in Scotland, and is properly confidered as one of the moft ufeful and excellent difcoveries of modern times. A table of thefe numbers was firft publifhed by him at Edinburgh, in the year 1614, in a treatife entitled *Canon Mirificum Logarithmorum*; and as their great utility and extenfive application were fufficiently apparent, they were immediately received by all the Learned throughout Europe. Mr. Henry Briggs, then profeffor of geometry at Grefham College, on hearing of the difcovery, fet out upon a vifit to the noble inventor, and foon afterwards they jointly undertook the arduous tafk of computing new tables on this fubject, and reducing them to a more convenient form than that which was at firft thought of. But Lord Napier dying before they were finifhed, the whole burden fell upon Mr. Briggs, who, with prodigious labour and great fkill, made an entire Canon, according to the new form, for all numbers from 1 to 20000, and from 9000 to 10100, to 14 places of figures, and publifhed it at London in the year 1624, in a treatife entitled Arithmetica Logarithmica, with directions for fupplying the intermediate chiliads.

Thus $\left\{\begin{array}{lllllll} 0, & 1, & 2, & 3, & 4, & 5, & 6, \text{ Indices, or logarithms.} \\ 1, & 2, & 4, & 8, & 16, & 32, & 64, \text{ Geometric progreffion.} \end{array}\right.$

Or $\left\{\begin{array}{lllllll} 0, & 1, & 2, & 3, & 4, & 5, & 6, \quad \text{Indices, or logarithms} \\ 1, & 3, & 9, & 27, & 81, & 243, & 729, \text{ Geometric progreffion} \end{array}\right.$

Or $\left\{\begin{array}{lllllll} 0, & 1, & 2, & 3, & 4, & 5, & \text{Indices, or logs.} \\ 1, & 10, & 100, & 1000, & 10000, & 100000, & \text{Geom. progref.} \end{array}\right.$

Where it is evident, that the fame indices ferve equally for any geometric feries ; and confequently there may be an endlefs variety of fyftems of logarithms, to the fame common numbers, by only changing the fecond term, 2, 3, or, 10, &c, of the geometrical feries of whole numbers ; and

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This Canon was again publifhed in Holland by Adrian Vlacq, anno 1628, together with the Logarithms of all the numbers which Mr. Briggs had omitted ; but he contracted them down to 10 places of decimals. Mr. Briggs alfo computed the Logarithms. of the fines, tangents and fecants, to every degree, and centefm, or 100th part of a degree, of the whole quadrant ; and fubjoined them to the natural fines, tangents, and fecants, which he had before computed to 15 places of figures. Thefe Tables, together with their conftruction and ufe. were firft publifhed in the year 1633, after Mr. Briggs's death, by Mr. Henry Gellibrand, under the title of Trigonometria Britannica.

Benjamin Urfinus alfo gave a Table of Logarithms to every 10 feconds. And Chr. Wolf, in his Mathematical Lexicon, fays that one Van Lofer had computed them to every fingle fecond, but his untimely death prevented their publication.

A great number of other authors have treated on this fubject ; but as their numbers are frequently inaccurate and incommodioufly difpofed, they are now generally neglected. The Tables in moft repute at prefent, are thofe of Gardiner in 4to, firft publifhed in the year 1742 ; and my own Tables in 8vo, firft printed in the year 1785, where the Logarithms of all numbers may be eafily found from 1 to 10000000 ; and thofe of the fines, tangents, and fecants, to any degree of accuracy required.

Alfo, Mr. Michael Taylor's Tables in large 4to, containing the common logarithms, and the logarithmic fines and tangents to every fecond of the quadrant. And, in France, the new book of logarithms by Callet ; the 2d edition of which, in 1795, has the tables ftill farther extended, and are printed with what they call ftereotypes, the types in each page being foldered together into a folid mafs or block.

Dodfon's Antilogarithmic Canon is likewife a very ingenious work. and of great ufe for finding the numbers anfwering to any given logarithm.

by

by interpolation the whole fyftem of numbers may be made to enter the geometric feries, and receive their proportional logarithms, whether integers or decimals.

It is alfo apparent, from the nature of thefe feries, that if any two indices be added together, their fum will be the index of that number which is equal to the product of the two terms, in the geometric progreffion, to which thofe indices belong.

Thus, the indices 2 and 3, being added together, make 5; and the numbers 4 and 8, or the terms correfponding to thofe indices, being multiplied together, make 32, which is the number anfwering to the index 5.

In like manner, if any one index be fubtracted from another, the difference will be the index of that number which is equal to the quotient of the two terms to which thofe indices belong.

Thus, the index 6, minus the index 4, is $= 2$; and the terms correfponding to thofe indices are 64 and 16, whofe quotient is $= 4$; which is the number anfwering to the index 2.

For the fame reafon, if the logarithm of any number be multiplied by the index of its power, the product will be equal to the logarithm of that power.

Thus, the index or logarithm of 4, in the above feries, is 2; and if this number be multiplied by 3, the product will be $= 6$; which is the logarithm of 64, or the third power of 4.

And, if the logarithm of any number be divided by the index of its root, the quotient will be equal to the logarithm of that root.

Thus, the index or logarithm of 64 is 6; and if this number be divided by 2, the quotient will be $= 3$; which is the logarithm of 8, or the fquare root of 64.

The logarithms moft convenient for practice, are fuch as are adapted to a geometric feries increafing in a tenfold proportion, as in the laft of the above forms; and are thofe which are to be found, at prefent, in moft of the common tables on this fubject.

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The diftinguifhing mark of this fyftem of logarithms is, that the index or logarithm of 10 is 1; that of 100 is 2; that of 1000 is 3, &c. And, in decimals, the logarithm of ·1 is — 1; that of ·01 is — 2; that of ·001 is — 3; &c. The log. of 1 being 0 in every fyftem.

From whence it follows, that the logarithm of any number between 1 and 10, muft be 0 and fome fractional parts; and that of a number between 10 and 100, will be 1 and fome fractional parts; and fo on, for any other number whatever.

And fince the integral part of a logarithm is always thus readily found, it is ufually called the Index, or Characteriftic; and is commonly omitted in the tables; being left to be fupplied by the operator himfelf, as occafion requires.

Another definition of Logarithms is, that the logarithm of any number is the index of that power of fome other number, which is equal to the given number. So if there be $N = r^n$, then $n$ is the log. of N; where $n$ may be either pofitive or negative, or nothing, and the root $r$ any number whatever, according to the different fyftems of logarithms.

When $n$ is $= 0$, then N is $= 1$, whatever the value of $r$ is; which fhews, that the log. of 1 is always 0, in every fyftem of logarithms.

When $n$ is $= 1$, then N is $= r$; fo that the radix $r$ is always that number whofe log. is 1, in every fyftem.

When the radix $r$ is $= 2·718281828459$ &c, the indices $n$ are the hyperbolic or Napier's log. of the numbers N; fo that $n$ is always the hyp. log. of the number N or $\overline{2·718, \&c.}|^n$.

But when the radix $r$ is $= 10$, then the index $n$ becomes the common or Briggs's log. of the number N; fo that the common log. of any number $10^n$ or N, is $n$ the index of that power of 10 which is equal to the faid number. Thus 100, being the fecond power of 10, will have 2 for its logarithm; and 1000, being the third power of 10, will have 3 for its logarithm: hence alfo, if 50 be $= 10^{1·69897}$, then is 1·69897 the common log. of 50. And, in general, the following decuple feries of terms,

viz. $10^4$, $10^3$, $10^2$, $10^1$, $10^0$ $10^{-1}$ $10^{-2}$, $10^{-3}$, $10^{-4}$,

or 10000, 1000, 100, 10, 1, ·1, ·01, ·001, ·0001,

have 4, 3, 2, 1, 0, —1, —2, —3, —4,

for their logarithms, refpectively. And from this fcale of numbers and logarithms, the fame properties eafily follow, as above mentioned.

## PROBLEM.

*To compute the Logarithm to any of the Natural Numbers*
1, 2, 3, 4, 5, *&c.*

### RULE I*.

1. TAKE the geometrical feries, 1, 10, 100, 1000, 10000, &c, and apply to it the arithmetical feries, 0, 1, 2, 3, 4, &c, as logarithms.

2. Find a geometric mean between 1 and 10, 10 and 100, or any other two adjacent terms of the feries, betwixt which the number propofed lies.

3. Between the mean, thus found, and the neareft extreme, find another geometrical mean, in the fame manner; and fo on, till you are arrived within the propofed limit of the number whofe logarithm is fought.

4. Find as many arithmetical means, in the fame order as you found the geometrical ones, and the laft of thefe will be the logarithm anfwering to the number required.

### EXAMPLE.

Let it be required to find the logarithm of 9

Here the propofed number lies between 1 and 10.

Firft, then, the log. of 10 is 1, and the log. of 1 is 0;

theref. $\overline{1 + 0} \div 2 = \frac{1}{2} = \cdot 5$ is the arithmetical mean, and $\sqrt{10 \times 1} = \sqrt{10} = 3 \cdot 1622777$ the geom. mean; hence the log. of $3 \cdot 1622777$ is 5.

Secondly, the log. of 10 is 1, and the log. of $3 \cdot 1622777$ is $\cdot 5$; theref. $\overline{1 + \cdot 5} \div 2 = \cdot 75$ is the arithmetical mean, and $\sqrt{10 \times 3 \cdot 1622777} = 5 \cdot 6234132$ is the geom. mean; hence the log. of $5 \cdot 6234132$ is $\cdot 75$.

Thirdly, the log. of 10 is 1, and the log. of $5 \cdot 6234132$ is $\cdot 75$; theref. $\overline{1 + \cdot 75} \div 2 = \cdot 875$ is the arithmetical mean, and $\sqrt{10 \times 5 \cdot 0234132} = 7 \cdot 4989422$ the geom. mean; hence the log. of $7 \cdot 4989422$ is $\cdot 875$.

Fourthly, the log. of 10 is 1, and the log. of $7 \cdot 4989422$ is $\cdot 875$; theref. $\overline{1 + \cdot 875} \div 2 = \cdot 9375$ is the arithmetical mean, and $\sqrt{10 \times 7 \cdot 4989421} = 8 \cdot 6596431$ the geom. mean; hence the log. of $8 \cdot 6596431$ is $\cdot 9375$.

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* The reader who wifhes to inform himfelf more particularly concerning the hiftory, nature, and conftruction of Logarithms, may confult the introduction to my Mathematical Tables, lately publifhed, where he will find his curiofity gratified.

Fifthly,

Fifthly, the log. of 10 is 1, and the log. of 8·6596431 is ·9375; theref. $1 + ·9375 \div 2 = ·96875$ is the arithmetical mean, and $\sqrt{10 \times 8·6596431} = 9·3057204$ the geom. mean: hence the log. of 9·3057204 is ·96875.

Sixthly, the log. of 8·6596431 is ·9375, and the log. of 9·3057204 is ·96875; theref. $·9375 + ·96875 \div 2 = ·953125$ is the arith. mean, and $\sqrt{8·6596431 \times 9·3057204} = 8·9768713$ the geometric mean: hence the log. of 8·9768713 is ·953125.

And proceeding in this manner, after 25 extractions, it will be found that the logarithm of 8·9999998 is ·9542425; which may be taken for the logarithm of 9, as it differs so little from it, that it is sufficiently exact for all practical purposes.

And in this manner were the logarithms of almost all the prime numbers at first found.

### RULE II*.

LET $b$ be the number whose logarithm is required to be found; and $a$ the number next less than $b$, so that $b - a = 1$, the logarithm of $a$ being known; and let $s$ denote the sum of the two numbers $a + b$. Then

1. Divide the constant decimal ·8685889638 &c, by $s$, and reserve the quotient: divide the reserved quotient by the square of $s$, and reserve this quotient: divide this last quotient also by the square of $s$, and again reserve the quotient: and thus proceed, continually dividing the last quotient by the square of $s$, as long as division can be made.

2. Then write these quotients orderly under one another, the first uppermost, and divide them respectively by the odd numbers 1, 3, 5, 7, 9, &c, as long as division can be made; that is, divide the first reserved quotient by 1, the second by 3, the third by 5, the fourth by 7, and so on.

3. Add all these last quotients together, and the sum will be the logarithm of $b \div a$; therefore to this logarithm add also the given logarithm of the said next less number $a$, so will the last sum be the logarithm of the number $b$ proposed.

---

* For the demonstration of this rule, see the Mathematical Tables, pa. 128, &c.

That is,

$$\text{Log. of } b \text{ is log. } a + \frac{n}{s} \times \left(1 + \frac{1}{3s^2} + \frac{1}{5s^4} + \frac{1}{7s^6} + \&c.\right)$$

where $n$ denotes the conftant given decimal ·8685889638 &c.

### EXAMPLES.

Ex. 1. Let it be required to find the log. of the number 2. Here the given number $b$ is 2, and the next lefs number $a$ is 1, whofe log. is 0; alfo the fum $2 + 1 = 3 = s$, and its fquare $s^2 = 9$. Then the operation will be as follows :

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3 ) | ·868588965 | | 1 ) | ·289529654 ( | ·289529654 |
| 9 ) | ·289529654 | | 3 ) | 32169962 ( | 10723321 |
| 9 ) | 32169962 | | 5 ) | 3574440 ( | 714888 |
| 9 ) | 3574440 | | 7 ) | 397160 ( | 56737 |
| 9 ) | 397160 | | 9 ) | 44129 ( | 4903 |
| 9 ) | 44129. | | 11 ) | 4903 ( | 446 |
| 9 ) | 4903 | | 13 ) | 545 ( | 42 |
| 9 ) | 545 | | 15 ) | 61 ( | 4 |
| 9 ) | 61 | | | | |

log. of $\frac{2}{1}$ - - ·301029995
add log. 1 - 000000000

log. of 2 - ·301029995

Ex. 2. To compute the logarithm of the number 3. Here $b = 3$, the next lefs number $a = 2$, and the fum $a + b = 5 = s$, whofe fquare $s^2$ is 25, to divide by which, always multiply by ·04. Then the operation is as follows :

| | | | | | |
|---|---|---|---|---|---|
| 5 ) | ·868588964 | | 1 ) | ·173717793 ( | ·173717793 |
| 25 ) | 173717793 | | 3 ) | 6948712 ( | 2316237 |
| 25 ) | 6948712 | | 5 ) | 277948 ( | 55590 |
| 25 ) | 277948 | | 7 ) | 11118 ( | 1588 |
| 25 ) | 11118 | | 9 ) | 445 ( | 50 |
| 25 ) | 445 | | 11 ) | 18 ( | 2 |
| | 18 | | | | |

log. of $\frac{3}{2}$ - - ·176091260
log. of 2 add - ·301029995

log. of 3 fought ·477121255

Ex. 3. To find the log. of 7 ; that of 6 being ·77815125.

Then, becaufe the fum of the logarithms of numbers, gives the logarithm of their product ; and the difference of the logarithms, gives the logarithm of the quotient of the numbers ;

numbers ; from the above two logarithms, and the logarithm of 10, which is 1, we may raife a great many logarithms, as in the following examples :-

**EXAMPLE 4.**

Becaufe $2 \times 2 = 4$, therefore

to log. 2  -  $\cdot3010299995\frac{2}{3}$

add log. 2  -  $\cdot3010299995\frac{2}{3}$

fum is log. 4   $\cdot6020599991\frac{1}{3}$

**EXAMPLE 7.**

Becaufe $3^2 = 9$, therefore

log. 3  -  $\cdot477121254\frac{7}{10}$

mult. by 2        2

gives log. 9 $\cdot954242509$

**EXAMPLE 5.**

Becaufe $2 \times 3 = 6$, therefore

to log. 2  -  $\cdot301029995$

add log. 3   $477121255$

fum is log. 6 $\cdot778151250$

**EXAMPLE 8.**

Becaufe $\frac{10}{2} = 5$, therefore

from log. 10 $1\cdot000000000$

take log. 2   $\cdot3010299995\frac{2}{3}$

leaves log 5   $\cdot6989700004\frac{1}{3}$

**EXAMPLE 6.**

Becaufe $2^3 = 8$, therefore

log. 2  -  $\cdot3010299995\frac{2}{3}$

mult. by 3        3

gives log. 8 $\cdot903089987$

**EXAMPLE 9.**

Becaufe $3 \times 4 = 12$, therefore

to log. 3  -  $\cdot477121255$

add log. 4     $\cdot602059991$

gives log. 12 $1\cdot079181246$

And thus, computing, by this general rule, the logarithms to the other prime numbers 7, 11, 13, 17, 19, 23, &c, and then ufing compofition and divifion, we may eafily find as many logarithms as we pleafe, or may fpeedily examine any logarithm in the table*.

---

* There are, befides thefe, many other ingenious methods, which later writers have difcovered for finding the logarithms of numbers, in a much eafier way than by the original inventor ; but, as they cannot be underftood without a knowledge of fome of the higher branches of the mathematics, it is thought proper to omit them, and to refer the reader to thofe works which are written exprefsly on the fubject.

It would likewife much exceed the limits of this compendium, to point out all the peculiar artifices that are made ufe of for con-ftructing an entire table of thefe numbers; fuch as thofe of Gardiner, Sherwin, and others, who have treated on this fubject ; but any information of this kind, which the learner may wifh to obtain, may be found in the Tables, before mentioned.

*Defcription and Ufe of the* TABLE *of* LOGARITHMS.

HAVING explained the method of making a table of the logarithms of numbers, greater than unity; the next thing to be done is, to fhew how the logarithms of fractional quantities may be found. And, in order to this, it may be obferved, that as in the former cafe a geometric feries is fuppofed to increafe towards the left, from unity, fo in the latter cafe it is fuppofed to decreafe towards the right hand, ftill beginning with unit; as exhibited in the general defcription, page 164, where the indices being made negative, ftill fhew the logarithms to which they belong. Whence it appears, that as $+$ 1 is the log. of 10, fo $-$ 1 is the log. of $\frac{1}{10}$ or ·1; and as $+$ 2 is the log. of 100, fo $-$ 2 is the log. of $\frac{1}{100}$ or ·01 : and fo on.

Hence it appears in general, that all numbers which confift of the fame figures, whether they be integral, or fractional, or mixed, will have the decimal parts of their logarithms the fame, but differing only in the index, which will be more or lefs, and pofitive or negative, according to the place of the firft figure of the number.

Thus, the logarithm of 2651 being 3·4234097, the log. of $\frac{1}{10}$, or $\frac{1}{100}$, or $\frac{1}{1000}$, &c, part of it; will be as follows:

| Numbers. | Logarithms. |
|---|---|
| 2 6 5 1 | 3 · 4 2 3 4 0 9 7 |
| 2 6 5·1 | 2 · 4 2 3 4 0 9 7 |
| 2 6·5 1 | 1 · 4 2 3 4 0 9 7 |
| 2·6 5 1 | 0 · 4 2 3 4 0 9 7 |
| ·2 6 5 1 | —1 · 4 2 3 4 0 9 7 |
| ·0 2 6 5 1 | —2 · 4 2 3 4 0 9 7 |
| ·0 0 2 6 5 1 | —3 · 4 2 3 4 0 9 7 |

From this it alfo appears, that the index, or characteriftic, of any logarithm, is always lefs by 1 than the number of integer figures which the natural number confifts of; or it is equal to the diftance of the firft or left-hand figure, from the place of units, or firft place of integers, whether on the left, or on the right of it : and this index is conftantly to be placed on the left hand fide of the decimal part of the logarithm.

When there are integers in the given number, the index is always affirmative; but when there are no integers, the index is negative, and is to be marked by a fhort line drawn before it, or elfe above it. Thus,

A number having  1, 2, 3, 4, 5, &c, integer places, the index of its log. is 0, 1, 2, 3, 4, &c, or 1 lefs than thofe places.

And

And a decimal fraction having its first figure in the 1ft, 2d, 3d, 4th, &c, place of decimals, has always —1, —2, —3, —4, &c, for the index of its logarithm,

It may alfo be obferved, that though the indices of fractional quantities are negative, yet the decimal parts of their logarithms are always affirmative.

### 1. TO FIND, IN THE TABLE, THE LOGARITHM TO ANY NUMBER.

1. *If the Number do not exceed* 100000, the decimal part of the logarithm is found, by infpection in the table, ftanding againft the given number, in this manner; viz. in moft tables, the firft four figures of the given number in the firft column of the page, and the fifth figure one of thofe along the top line of it ; then in the angle of meeting are the laft four figures of the logarithm, and the firft 3 figures of the fame at the beginning of the fame line : to which is to be prefixed the proper index, which is always 1 lefs than the number of integer figures.

So the logarithm of 34·092 is 1·5326525, that is, the decimal ·5326525 found in the table, with the index 1 prefixed, becaufe the given number contains two integers.

2. *But if the given number contain more than five figures ;* take out the logarithm of the firft five figures by infpection in the table as before, as alfo the next greater logarithm, fubtracting the one logarithm from the other, as alfo their correfponding numbers the one from the other. Then fay,

As the difference between the two numbers,
Is to the difference of their logarithms,
So is the remaining part of the given number,
To the proportional part of the logarithm.

Which part being added to the lefs logarithm, before taken out, gives the whole logarithm fought, very nearly.

## EXAMPLE.

To find the logarithm of the number 34.09264.
The log. of 3409200, as before, is 5326525,
And log. of 3409300 — is 5326652,

The diffs. are 100 and 127

Then, as 100 : 127 :: 64 : 81, the proportional part.
This added to — 5326525, the first log.
Gives, with the index, 1.5326606 for the log. of 34.09264.

Or, in the best tables, the proportional part may often be taken out by inspection, by means of the small tablets of proportional parts placed in the margin.

3. If the number consist both of integers and fractions, or is entirely fractional; find the decimal part of the logarithm the same as if all its figures were integral ; then this, being prefixed to the proper characteristic, will give the logarithm required.

4. And if the given number be a proper vulgar fraction ; subtract the logarithm of the denominator from the logarithm of the numerator, and the remainder will be the logarithm sought ; which, being that of a decimal fraction, must always have a negative index.

5. But if it be a mixed number ; reduce it to an improper fraction, and find the difference of the logarithms of the numerator and denominator, in the same manner as before.

## EXAMPLES.

1. To find the log. of $\frac{37}{97}$.

| | | |
|---|---|---|
| Log. of 37 | — | 1.5682017 |
| Log. of 97 | — | 1.9731279 |
| Dif. log. of $\frac{37}{97}$ | —1.5950738 | |
| Where the index 1 is negative. | | |

2. To find the log. of $17\frac{14}{23}$.

First, $17\frac{14}{23} = \frac{405}{23}$. Then,

| | | |
|---|---|---|
| Log. of 405 | — | 2.6074550 |
| Log. of 23 | — | 1.3617278 |
| Dif. log. of $17\frac{14}{23}$ | 1.2457272 | |

## II. TO FIND THE NATURAL NUMBER TO ANY GIVEN LOGARITHM.

This is to be found in the tables by the reverse method to the former, namely, by searching for the proposed logarithm among those in the table, and taking out the corresponding number by inspection, in which the proper number of integers are to be pointed out, viz. 1 more than the index. For, in finding the number answering to any given logarithm, the index always shews how far the first figure

must

muſt be removed from the place of units, viz. to the left hand, or integers, when the index is affirmative; but to the right hand, or decimals, when it is negative.

## EXAMPLES.

So, the number to the log. 1·5326525 is 34·092.

And the number of the log. 1̅·5326525 is ·34092.

But if the logarithm cannot be exaſtly found in the table: take out the next greater and the next lefs, fubtraſting the one of thefe logarithms from the other, as alfo their natural numbers the one from the other, and the lefs logarithm from the logarithm propofed. Then fay,

As the difference of the firſt or tabular logarithms,

Is to the difference of their natural numbers,

So is the differ. of the given log. and the leaſt tabular log.

To their correfponding numeral difference.

Which being annexed to the leaſt natural number above taken, gives the natural nnmber fought, correfponding to the propofed logarithm.

## EXAMPLE.

So, to find the natural number anfwering to the given logarithm 1·5326606.

Here the next greater and next lefs tabular logarithms, with their correfponding numbers, &c, are as below:

Next greater 5326652 its num. 3409300; given log. 5326606

Next lefs    5326525 its num. 3409200; next lefs 5326525

Differences      127 – –      100 – –            81

Then, as 127 : 100 :: 81 : 64 nearly, the numeral differ.

Therefore 34·09264 is the number fought, marking off two integers, becaufe the index of the given logarithm is 1.

Had the index been negative, thus 1̅·5326606, its corre-fponding number would have been ·3409264, wholly de-cimal.

Or the proportional numeral difference may be found, in the beſt tables, by infpeſtion, by means of the fmall tablets of proportional parts placed in the margins of the pages.

### III. MULTIPLICATION BY LOGARITHMS.

#### RULE.

TAKE out the logarithms of the factors from the table, then add them together, and their sum will be the logarithm of the product required. Then, by means of the table, take out the natural number anfwering to the fum, for the product fought.

Obferving to add what is to be carried from the decimal part of the logarithm to the affirmative index or indices, or elfe fubtract it from the negative.

Alfo, adding the indices together when they are of the fame kind, both affirmative or both negative ; but fubtract-ing the lefs from the greater, when the one is affirmative and the other negative, and prefixing the fign of the greater to the remainder.

#### EXAMPLES.

1. To multiply 23·14 by 5·062.

| Numbers. | Logs. |
|---|---|
| 23·14 – | 1·3643634 |
| 5·062 – | 0·7043221 |

Product 117·1347 2·0686855

2. To multiply 2·581926 by 3·457291.

| Numbers. | Logs. |
|---|---|
| 2·581926 – | 0·4119438 |
| 3·457291 – | 0·5387359 |

Prod. 8·92647 – 0·9506797

3. To mult. 3·902, and 597·16, and ·0314728 all together.

| Numbers. | Logs. |
|---|---|
| 3·902 – | 0·5912873 |
| 597·16 – | 2·7760907 |
| ·0314728 – | 2·4979353 |

Prod. 73·33533 – 1·8653133

Here the —2 cancels the 2, and the 1 to carry from the decimals is fet down.

4. To mult. 3·586, and 2·1046, and 0·8372, and 0·0294 all together.

| Numbers. | Logs. |
|---|---|
| 3·586 – | 0·5546103 |
| 2·1046 – | 0·3231696 |
| 0·8372 – | 1·9228292 |
| 0·0294 – | 2·4683473 |

Prod. 0·1857618 — 1·2689564

Here the 2 to carry cancels the —2, and there remains the —1 to fet down.

## DIVISION BY LOGARITHMS.

### RULE.

FROM the logarithm of the dividend subtract the logarithm of the divisor, and the number answering to the remainder will be the quotient required.

Observing to change the sign of the index of the divisor, from affirmative to negative, or from negative to affirmative; then take the sum of the indices if they be of the same name, or their difference when of different signs, with the sign of the greater, for the index to the logarithm of the quotient.

And also, when 1 is borrowed, in the left-hand place of the decimal part of the logarithm, add it to the index of the divisor when that index is affirmative, but subtract it when negative; then let the index arising from thence be changed, and worked with as before.

### EXAMPLES.

1. To divide 24163 by 4567.

| Numbers. | Logs. |
|---|---|
| Dividend 24163 — | 4·3831509 |
| Divisor    4567 — | 3·6596310 |
| | |
| Quot.    5·290782 | 0·7235199 |

2. To divide 37·149 by 523·76.

| Numbers. | Logs. |
|---|---|
| Dividend 37·149 — | 1·5699471 |
| Divisor   523·76 — | 2·7191323 |
| | |
| Quot. ·07092752 — | 2·8508148 |

3. Divide ·06314 by ·007241.

| Numbers. | Logs. |
|---|---|
| Divid. ·06314 | —2·8003046 |
| Divisor ·007241 | —3·8597985 |
| | |
| Quot. 8·719792 | 0·9405061 |

4. To divide ·7438 by 12·9476.

| Numbers. | Logs. |
|---|---|
| Divid.   ·7438 | —1·8714562 |
| Divisor 12·9476 | 1·1121893 |
| | |
| Quot. ·05744694 | —2·7592669 |

Here 1 carried from the decimals to the —3, makes it become —2, which taken from the other —2, leaves 0 remaining.

Here the 1 taken from the —1, makes it become —2, to set down.

---

Note. As to the Rule-of-Three, or Rule of Proportion, it is performed by adding the logarithms of the 2d and 3d terms, and subtracting that of the first term from their sum.

INVO-

## INVOLUTION BY LOGARITHMS.

### RULE.

TAKE out the logarithm of the given number from the table.

Multiply the log. thus found, by the index of the power propofed.

Find the number anfwering to the product, and it will be the power required.

*Note.* In multiplying a logarithm with a negative index, by an affirmative number, the product will be negative.

But what is to be carried from the decimal part of the logarithm, will always be affirmative.

And therefore their difference will be the index of the product, and is always to be made of the fame kind with the greater.

### EXAMPLES.

1. To fquare the number 2·5791.

| Numb. | | Log. |
|---|---|---|
| Root 2·5791 | – | 0·4114682 |
| The index | – – | 2 |
| Power 6·651756 | | 0·8229364 |

2. To find the cube of 3·07146.

| Numb. | | Log. |
|---|---|---|
| Root 3·07146 | – | 0·4873449 |
| The index | – – | 3 |
| Power 28·97575 | | 1·4620347 |

3. To raife ·09163 to the 4th power.

| Numb. | | Log. |
|---|---|---|
| Root ·09163 | | —2·9620377 |
| The index | – – | 4 |
| Po. ·00000704938 | | —5·8481508 |

Here, 4 times the negative index being — 8, and 3 to carry, the difference —5 is the index of the product.

4. To raife 1·0045 to the 365th power.

| Numb. | | Log. |
|---|---|---|
| Root 1·0045 | – | 0·0019499 |
| The index | – | 365 |
| | | 97495 |
| | | 116994 |
| | | 58497 |
| Power 5·148888 | | ·7117135 |

**Evo-**

## EVOLUTION BY LOGARITHMS.

TAKE the log. of the given number out of the table.
Divide the log. thus found, by the index of the root.
Then the number anfwering to the quotient, will be the
root.

*Note.* When the index of the logarithm, to be divided, is
negative, and does not exactly contain the divifor, without
fome remainder, increafe the index by fuch a number as
will make it exactly divifible by the index, carrying the units
borrowed, as fo many tens, to the left-hand place of the
decimal, and then divide as in whole numbers.

Ex. 1. To find the fquare root
of 365.

| Numb. | Log. |
|---|---|
| Power 365 | 2 ) 2·5622929 |
| Root 19·10498 | 1·2811465 |

Ex. 2. To find the 3d root of
12345.

| Numb. | Log. |
|---|---|
| Power 12345 | 3 ) 4·0914911 |
| Root 23·11162 | 1·3638304 |

Ex. 3. To find the 10th root
of 2.

| Numb. | Log. |
|---|---|
| Power 2 – | 10 ) 0·3010300 |
| Root 1·071773 | 0·0301030 |

Ex. 4. To find the 365th root
of 1·045.

| Numb. | Log. |
|---|---|
| Power 1·045 | 365)0·0191163 |
| Root 1·000121 | 0·0000524 |

Ex. 5. To find $\sqrt{·093}$.

| Numb. | Log. |
|---|---|
| Power ·093 | 2 ) —2·9684829 |
| Root ·304959 | —1·4842415 |

Here the divifor 2 is con-
tained exactly once in the
negative index — 2, and
therefore the index of the
quotient is —1.

Ex. 6. To find the $\sqrt[3]{·00048}$.

| Numb. | Log. |
|---|---|
| Power ·00048 | 3)—4·6812412 |
| Ro. ·07829735 | —2·8937471 |

Here, the divifor 3 not being exactly
contained in —4, it is augmented by 2,
to make up 6, in which the divifor is con-
tained juft 2 times; then the 2, thus bor-
rowed, being carried to the decimal fi-
gure 6, makes 26, which divided by
3, gives 8, &c.

Ex. 7. To find $\frac{1}{3}\sqrt{\frac{3}{7}} \times ·001 \sqrt[3]{\frac{9}{17}}$.

Ex. 8. To find $\dfrac{2\frac{3}{7}\sqrt{\frac{11}{27}} \times ·02\sqrt[5]{13\frac{3}{31}}}{7\frac{8}{111}\sqrt[3]{37\frac{1}{3}} \times ·09\sqrt[4]{\frac{17}{319}}}$.

Ex. 9. As $\sqrt{\frac{1}{7}} : \sqrt[3]{\frac{11}{784394}} :: \sqrt[5]{\frac{1}{4181631}} : ?$

Ex. 10. As $\sqrt{\frac{4}{1011079}} : \left(\frac{1}{1181}\right)^3 :: \sqrt[4]{\frac{1}{109476}} : ?$

ALGE-

# ALGEBRA.

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## DEFINITIONS AND NOTATION.

1. ALGEBRA is the art of computing by symbols. It is sometimes also called Analysis; and is a general kind of arithmetic, or universal way of computation.

2. In this science, quantities of all kinds are represented by the letters of the alphabet. And the operations to be performed with them, as addition or subtraction, &c, are denoted by certain simple characters, instead of being expressed by words at length.

3. In algebraical questions, some quantities are known or given, viz, those whose values are known : and others unknown, or are to be found out, viz, those whose values are not known. The former of these are represented by the leading letters of the alphabet, $a, b, c, d$, &c ; and the latter, or unknown quantities, by the final letters, $z, y, x, u$, &c.

4. The characters used to denote the operations, are chiefly the following :

$+$ signifies addition, and is named *plus*.

$-$ signifies subtraction, and is named *minus*.

$\times$ or $.$ signifies multiplication, and is named *into*.

$\div$ signifies division, and is named *by*.

$\sqrt{}$ signifies the square root ; $\sqrt[3]{}$ the cube root ; $\sqrt[4]{}$ the 4th root, &c ; and $\sqrt[n]{}$ the $n$th root.

$: :: :$ signifies proportion.

$=$ signifies equality, and is named *equal to*.

And so on for other operations.

Thus $a + b$ shews, that the number represented by $b$ is to be added to that represented by $a$.

$a - b$ shews, that the number represented by $b$ is to be subtracted from that represented by $a$.

$a \backsim b$ represents the difference of $a$ and $b$, when it is not known which is the greater.

$ab$, or $a \times b$, or $a.b$, denotes the product, by multiplication, of the numbers represented by $a$ and $b$.

$a \div b$, or $\dfrac{a}{b}$, ſhews that the number repreſented by $a$ is to be divided by that which is repreſented by $b$.

$\dfrac{a}{b}$ is the reciprocal of $\dfrac{b}{a}$, and $\dfrac{1}{a}$ the reciprocal of $a$.

$a : b :: c : d$ denotes that $a$ is in the ſame proportion to $b$, as $c$ is to $d$.

$x = a - b + c$ is an equation, ſhewing that $x$ is equal to the difference of $a$ and $b$, added to the quantity $c$.

$\sqrt{a}$, or $a^{\frac{1}{2}}$, is the ſquare root of $a$; $\sqrt[3]{a}$, or $a^{\frac{1}{3}}$, is the cube root of $a$; and $\sqrt[3]{a^2}$ or $a^{\frac{2}{3}}$ is the cube root of the ſquare of $a$; and $\sqrt[m]{a}$, or $a^{\frac{1}{m}}$, is the $m$th root of $a$; and $\sqrt[m]{a^n}$ or $a^{\frac{n}{m}}$ is the $n$th power of the $m$th root of $a$, or it is $a$ to the $\dfrac{n}{m}$ power.

$a^2$ is the ſquare of $a$; $a^3$ the cube of $a$; $a^4$ the fourth power of $a$; and $a^m$ the $m$th power of $a$.

$\overline{a + b} \times c$, or $(a + b)\,c$, is the product of the compound quantity $a + b$ multiplied by the ſimple quantity $c$. Uſing the bar ——, or the parentheſis ( ) as a vinculum, to connect ſeveral quantities into one.

$\overline{a + b} \div \overline{a - b}$, or $\dfrac{a + b}{a - b}$, expreſſed like a fraction, is the quotient of $a + b$ divided by $a - b$.

$\sqrt{ab + cd}$, or $(ab + cd)^{\frac{1}{2}}$, is the ſquare root of the compound quantity $ab + cd$. And $c\sqrt{ab + cd}$, or $c\,(ab + cd)^{\frac{1}{2}}$ denotes the product of $c$ into the ſquare root of the compound quantity $ab + cd$.

$\overline{a + b - c}^{3}$, or $(a + b - c)^3$, is the cube, or third power, of the quantity $a + b - c$.

$5a$ denotes that the quantity $a$ is to be taken 5 times, and $7 \cdot (b + c)$ is 7 times $b + c$. And theſe numbers, 5 or 7, ſhewing how often the quantities are to be taken, or multiplied, are called Co-efficients.

Alſo $\frac{1}{4}x$ denotes that $x$ is multiplied by $\frac{1}{4}$; thus $\frac{1}{4} \times x$ or $\frac{1}{4}x$.

5. Like Quantities, are thoſe which conſiſt of the ſame letters, and powers. As $a$ and $3a$; or $2ab$ and $4ab$; or $3a^2bc$ and $-5a^2bc$.

6. Unlike Quantities, are thoſe which conſiſt of different letters, or different powers. As $a$ and $b$; or $2a$ and $a^2$; or $3ab^2$ and $3abc$.

7. Simple Quantities, are thofe which confift of one term only. As $3a$, or $5ab$, or $6abc^2$.

8. Compound Quantities, are thofe which confift of two or more terms. As $a + b$, or $2a - 3c$, or $a + 2b - 3c$.

9. And when the compound quantity confifts of two terms, it is called a Binomial, as $a + b$; when of three terms, it is a Trinomial, as $a + 2b - 3c$; when of four terms, a Quadrinomial, as $2a - 3b + c - 4d$; and fo on. Alfo, a Multinomial or Polynomial, confifts of many terms.

10. A Refidual Quantity, is a binomial having one of the terms negative. As $a - 2b$.

11. Pofitive or Affirmative Quantities, are thofe which are to be added, or have the fign $+$. As $a$ or $+ a$, or $ab$: for when a quantity is found without a fign, it is underftood to be pofitive, or to have the fign $+$ prefixed.

12. Negative Quantities, are thofe which are to be fubtracted. As $- a$, or $- 2ab$, or $- 3ab^2$.

13. Like Signs, are either all pofitive $( + )$, or all negative $( - )$.

14. Unlike Signs, are when fome are pofitive $( + )$, and others negative $( - )$.

15. The Co-efficient of any quantity, as fhewn above, is the number prefixed to it. As 3, in the quantity $3ab$.

16. The Power of a quantity $(a)$, is its fquare $(a^2)$, or cube $(a^3)$, or biquadrate $(a^4)$, &c; called alfo, the 2d power, or 3d power, or 4th power, &c.

17. The Index or Exponent, is the number which denotes the power or root of a quality. So 2 is the exponent of the fquare or fecond power $a^2$; and 3 is the index of the cube or 3d power; and $\frac{1}{2}$ is the index of the fquare root, $a^{\frac{1}{2}}$ or $\sqrt{a}$; and $\frac{1}{3}$ is the index of the cube root, $a^{\frac{1}{3}}$ or $\sqrt[3]{a}$.

18. A Rational Quantity, is that which has no radical fign $( \sqrt{} )$ or index annexed to it. As $a$, or $3ab$.

19. An Irrational Quantity, or Surd, is that which has not an exact root, or is exprefled by means of the radical fign $\sqrt{}$. As $\sqrt{2}$, or $\sqrt{a}$, or $\sqrt[3]{a^2}$, or $ab^{\frac{1}{2}}$.

20. The Reciprocal of any quantity, is that quantity inverted, or unity divided by it. So, the reciprocal of $a$, or $\frac{a}{1}$, is $\frac{1}{a}$, and the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$.

21. The

21. The letters by which any fimple quantity is expreffed, may be ranged according to any order at pleafure. So the product of $a$ and $b$, may be either expreffed by $ab$, or $ba$; and the product of $a$, $b$, and $c$, by either $abc$, or $acb$, or $bac$, or $bca$, or $cab$, or $cba$; as it matters not which quantities are placed or multiplied firft. But it will be fometimes found convenient, in long operations, to place the feveral letters according to their order in the alphabet, as $abc$, which order alfo occurs moft eafily or naturally to the mind:

22. Likewife, the feveral members, or terms, of which a compound quantity is compofed, may be difpofed in any order at pleafure, without altering the value of the fignification of the whole. Thus, $3a - 2b + 4abc$ may alfo be written $3a + 4abc - 2ab$, or $4abc + 3a - 2ab$, or $-2ab + 3a + 4abc$, &c; for all thefe reprefent the fame thing, namely, the quantity which remains, when the quantity or term $2ab$ is fubtracted from the fum of the terms or quantities $3a$ and $4abc$. But it is moft ufual and natural, to begin with a pofitive term, and with the firft letters of the alphabet.

### SOME EXAMPLES FOR PRACTICE.

In finding the numeral values of various expreffions, or combinations, of quantities.

Suppofing $a = 6$, and $b = 5$, and $c = 4$, and $d = 1$, and $e = 0$. Then

1. Will $a^2 + 3ab - c^2 = 36 + 90 - 16 = 110$.

2. And $2a^3 - 3a^2b + c^3 = 432 - 540 + 64 = -44$.

3. And $a^2 \times \overline{a + b} - 2abc = 36 \times 11 - 240 = 156$.

4. And $\dfrac{a^3}{a + 3c} + c^2 = \dfrac{216}{18} + 16 = 12 + 16 = 28$.

5. And $\sqrt{2ac + c^2}$ or $\overline{2ac + c^2}|^{\frac{1}{2}} = \sqrt{64} = 8$.

6. And $\sqrt{c} + \dfrac{2bc}{\sqrt{2ac + c^2}} = 2 + \dfrac{40}{8} = 7$.

7. And $\dfrac{a^2 - \sqrt{b^2 - ac}}{2a - \sqrt{b^2 + ac}} = \dfrac{36 - 1}{12 - 7} = \dfrac{35}{5} = 7$.

8. And $\sqrt{b^2 - ac} + \sqrt{2ac + c^2} = 1 + 8 = 9$.

9. And $\sqrt{b^2 - ac + \sqrt{2ac + c^2}} = \sqrt{25 - 24 + 8} = 3$.

10. And $a^2b + c - d =$

11. And $5ab - 10b^2 + e =$

12. And

12. And $\dfrac{a^2 b}{c} \times d =$

13. And $\dfrac{a+b}{c} \times \dfrac{b}{d} =$

14. And $\dfrac{a+b}{c} - \dfrac{a-b}{d} =$

15. And $\dfrac{a^2 b}{c} + e =$

16. And $\dfrac{a^2 b}{c} \times e =$

17. And $\overline{b-c} \times \overline{d-e} =$

18. And $\overline{a+b} - \overline{c-d} =$

19. And $\overline{a+b} - c - d =$

20. And $a^2 c \times d^3 =$

21. And $acd - d =$

22. And $a^2 e + b^2 e + d =$

23. And $\dfrac{b-e}{d-e} \times \dfrac{a+b}{c-d} =$

24. And $\sqrt{a^2 + b^2} - \sqrt{a^2 - b^2} =$

25. And $3ac^2 + \sqrt[3]{a^3 - b^3} =$

26. And $4a^2 - 3a\sqrt{a^2 - ab\frac{2}{3}} =$

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## ADDITION.

ADDITION, in Algebra, is the connecting the quantities together by their proper signs, and incorporating or uniting into one term or sum, such as are similar, and can be united. As $3a + 2b - 2a = a + 2b$, the sum.

The rule of addition in algebra, may be divided into three cases ; one, when the quantities are like, and their signs like also ; a second, when the quantities are like, but their signs unlike ; and the third, when the quantities are unlike. Which are performed as follows*.

CASE

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* The reasons on which these operations are founded, will readily appear, by a little reflection on the nature of the quantities to be added,

## CASE I.

*When the Quantities are Like, and have Like Signs :*

FIRST set down the common sign ; after which set the sum of the co-efficients, found by adding them together ; to which annex the common letter or letters of the like quantities.

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added, or collected together. For, with regard to the first example, where the quantities are $3a$ and $5a$, whatever $a$ represents in the one term, it will represent the same thing in the other; so that 3 times any thing and 5 times the same thing, collected together, must needs make 8 times that thing. As if $a$ denote a shilling; then $3a$ is 3 shillings, and $5a$ is 5 shillings, and their sum 8 shillings. In like manner, $-2ab$ and $-7ab$, or $-2$ times any thing, and $-7$ times the same thing, make $-9$ times that thing.

As to the second case, in which the quantities are like, but the signs unlike; the reason of its operation will easily appear, by reflecting, that addition means only the uniting of quantities together by means of the arithmetical operations denoted by their signs $+$ and $-$, or of addition and subtraction; which being of contrary or opposite natures, the one co-efficient must be subtracted from the other, to obtain the incorporated or united mass.

As to the third case, where the quantities are unlike, it is plain that such quantities cannot be united into one, or otherwise added, than by means of their signs : thus, for example, if $a$ be supposed to represent a crown, and $b$ a shilling; then the sum of $a$ and $b$ can be neither $2a$ nor $2b$, that is, neither 2 crowns nor 2 shillings, but only 1 crown plus 1 shilling, that is $a + b$.

In this rule, the word *addition* is not very properly used ; being much too scanty to express the operation here performed. The business of this operation is to incorporate into one mass, or algebraic expression, different algebraic quantities; as far as an actual incorporation or union is possible ; and to retain the algebraic marks for doing it, in cases where the former is not possible. When we have several quantities, some affirmative and some negative ; and the relation of these quantities can in the whole or in part be discovered ; such incorporation of two or more quantities into one, is plainly effected by the foregoing rules.

It may seem a paradox, that what is called addition in algebra, should sometimes mean addition, and sometimes subtraction. But the paradox wholly arises from the scantiness of the name given to the algebraic process ; from employing an old term in a new and more enlarged sense. Instead of addition, call it incorporation, or union, or striking a balance, or any name to which a more extensive idea may be annexed, than that which is usually implied by the word addition; and the paradox vanishes.

Thus,

Thus, $3a$ added to $5a$, makes $8a$.

And — $2ab$ added to — $7ab$, makes — $9ab$.

And $5a + 7b$ added to $7a + 3b$, makes $12a + 10b$.

### OTHER EXAMPLES FOR PRACTICE.

| | | |
|---|---|---|
| 5 $a$ | — 6 $bx$ | 8 $bxy$ |
| 7 $a$ | — 3 $bx$ | 7 $bxy$ |
| 8 $a$ | — 2 $bx$ | 3 $bxy$ |
| 10 $a$ | — 7 $bx$ | 4 $bxy$ |
| 2 $a$ | — $bx$ | 5 $bxy$ |
| $a$ | — 5 $bx$ | $bxy$ |
| 33 $a$ | — 24 $bx$ | 28 $bxy$ |

| | | |
|---|---|---|
| 2 $y$ | 5 $x^2 + 5 xy$ | 7 $ax - 4y$ |
| 5 $y$ | 3 $x^2 + 2 xy$ | 8 $ax - 3y$ |
| 7 $y$ | $x^2 + 3 xy$ | 6 $ax - 2y$ |
| 4 $y$ | 7 $x^2 + 8 xy$ | 4 $ax - 3y$ |
| 3 $y$ | $x^2 + xy$ | 2 $ax - 2y$ |
| 21$y$ | 17$x^2 + 19xy$ | 27$ax - 14y$ |

| | | |
|---|---|---|
| 6 $xy$ | — 2 $y^2$ | 5 $a - 4 b$ |
| 15 $xy$ | — 8 $y^2$ | 7 $a - 6 b$ |
| 2 $xy$ | — 7 $y^2$ | 4 $a - 3 b$ |
| 7 $xy$ | — $y^2$ | 2 $a - 8 b$ |
| $1\frac{1}{2}$ $xy$ | — 6 $y^2$ | 6 $a - b$ |
| $\frac{1}{2}$ $xy$ | — $y^2$ | 3 $a - 2 b$ |

| | | |
|---|---|---|
| 20 — 15 $x^{\frac{1}{2}} - 2 xy$ | 7 $xy - 5 x + 3 ab$ |
| 35 — 13 $x^{\frac{1}{2}} - 4 xy$ | 3 $xy - x + 2 ab$ |
| 18 — 12 $x^{\frac{1}{2}} - 3 xy$ | 2 $xy - 3 x + 2 ab$ |
| 12 — 14 $x^{\frac{1}{2}} - 8 xy$ | 2 $xy - 4 x + 8 ab$ |
| 10 — 28 $x^{\frac{1}{2}} - 2 xy$ | 5 $xy - 3 x + ab$ |

### CASE II.

*When the Quantities are Like, but have Unlike Signs:*

ADD all the affirmative co-efficients into one sum, and all the negative ones into another, when there are several of a kind.

Subtract the less of these sums from the greater, and to the difference prefix the sign of the greater, and subjoin the common quantity.

So   $+ 5a$ and $- 3a$, united, make $+ 2a$.
And $- 5a$ and $+ 3a$, united, make $- 2a$.

#### OTHER EXAMPLES FOR PRACTICE.

| | | | |
|---|---|---|---|
| $- 3\,a$ | $+ 8\,ax^2$ | $+ 6\,x^3$ | $+ 8y$ |
| $+ 7\,a$ | $+ 7\,ax^2$ | $- 3\,x^3$ | $+ 7y$ |
| $+ 8\,a$ | $- 3\,ax^2$ | $- 13\,x^3$ | $+ 8y$ |
| $-\quad a$ | $- 4\,ax^2$ | $+ 2\,x^3$ | $- 3y$ |
| $- 2\,a$ | $+ 4\,ax^2$ | $+\quad x^3$ | $-\quad y$ |
| $+ 9\,a$ | $+ 12\,ax^2$ | $- 7\,x^3$ | $+ 19y$ |

| | | | |
|---|---|---|---|
| $- 2\,a^2$ | $+ 8\cdot b^2y^3$ | $- 3\,ab$ | $+\ 7$ |
| $- 3\,a^2$ | $+ 6\,b^2y^3$ | $+ 3\,ab$ | $- 10$ |
| $- 8\,a^2$ | $- 10\,b^2y^3$ | $+ 3\,ab$ | $-\ 6$ |
| $+ 10\,a^2$ | $- 20\,b^2y^3$ | $-\quad ab$ | $+\ 2$ |
| $+ 13\,a^2$ | $-\quad b^2y^3$ | $- 2\,ab$ | $+ 11$ |

| | | | |
|---|---|---|---|
| $- 2ax^{\frac{1}{2}}$ | $- 6\sqrt{ax}$ | $- 2y$ | $+ 2ax^{\frac{1}{2}}$ |
| $+\quad ax^{\frac{1}{2}}$ | $+ 2\sqrt{ax}$ | $+\ y$ | $+\quad ax^{\frac{1}{2}}$ |
| $- 3ax^{\frac{1}{2}}$ | $- 6\sqrt{ax}$ | $- 7y$ | $- 3ax^{\frac{1}{2}}$ |
| $+ 7ax^{\frac{1}{2}}$ | $+ 10\sqrt{ax}$ | $+ 5y$ | $+ 3ax^{\frac{1}{2}}$ |

## C A S E   III.

### *When the Quantities are Unlike.*

HAVING collected together all the like quantities, as in the two foregoing cafes, fet down thofe that are unlike, one after another, with their proper figns.

### E X A M P L E S.

$$5xy \qquad\qquad 2xy - 10x^2 \qquad\qquad 2ax - 150 + 2x^{\frac{1}{2}}$$
$$4ax \qquad\qquad -3x^2 + xy \qquad\qquad 3x^2 + 2ax + 6x^2$$
$$-xy \qquad\qquad -8x^2 - xy \qquad\qquad 5xy - 3x^{\frac{1}{2}} + 50$$
$$-4ax \qquad\qquad -xy + 9x^2 \qquad\qquad \sqrt{x} + 100 - 5x^2$$
$$\overline{\qquad 4xy \qquad} \qquad \overline{\quad xy - 12x^2 \quad} \qquad \overline{4ax + 4x^2 + 5xy}$$

$$6x^2y^2 \qquad\qquad 12ax - x^2 \qquad\qquad 6 + 10\sqrt{ax} - 3y$$
$$-4x^2y \qquad\qquad 4ax + xy \qquad\qquad x + 4\sqrt{xy} + 3y$$
$$-2axy \qquad\qquad 3y^2 - ax \qquad\qquad y - 2\sqrt{ax} - 3y$$
$$-3x^2y \qquad\qquad 2x^2 - 24 \qquad\qquad 20 + 3\sqrt{ax} - 3y$$

$$3x^2y \qquad\qquad 2\sqrt{x} - 8y \qquad\qquad a^2 - 8 + x^{\frac{1}{2}} - 2$$
$$-2xy^2 \qquad\qquad 3\sqrt{xy} + 10x \qquad\qquad a - 10 + a^2 - x$$
$$-3y^2x \qquad\qquad 2x + \sqrt{x} + y \qquad\qquad x^2 - a^2 + 8 - 4$$
$$-8x^2y \qquad\qquad -8 + \sqrt{x}y \qquad\qquad 10 - a - x^2 - y$$

Add $a + b$ and $3a - 5b$ together.
Add $5a - 8x$ and $3a - 4x$ together.
Add $6x - 5b + a + 8$ to $-5a - 4x + 4b - 3$.
Add $a + 2b - 3c - 10$ to $3b - 4a + 5c + 10$ and $5b - c$.
Add $a + b$ and $a - b$ together.
Add $3a + b - 10$ to $c - d - a$ and $-4c + 2a - 3b - 7$.
Add $3a^2 + b^2 - c$ to $2ab - 3a^2 + bc - b$.
Add $a^3 + b^2c - b^2$ to $ab^2 - abc + b^2$.
Add $9a - 8b + 10x - 6d - 7c + 50$ to $2x - 3a - 5c + 4b + 6d - 10$.

SUB-

## SUBTRACTION.

### RULE*.

SET down in one line the firſt quantities from which the ſubtraction is to be made; and underneath them place all the other quantities compoſing the ſubtrahend; ranging the like quantities under each other, as in Addition.

Then change all the ſigns ( + and — ) of the lower line, or conceive them to be changed; after which, collect all the terms together as in the caſes of Addition.

### EXAMPLES.

$$5a^2 - 2b$$
$$2a^2 - 5b$$
$$\overline{\phantom{xxxx}}$$
$$3a^2 + 3b$$

$$6x^2 - 8y + 3$$
$$2x^2 + 9y - 2$$
$$\overline{\phantom{xxxx}}$$
$$4x^2 - 17y + 5$$

$$5xy - 2 + 8x - y$$
$$3xy - 8 - 8x - 3y$$
$$\overline{\phantom{xxxx}}$$
$$2xy + 6 + 16x + 2y$$

$$3xy - 8$$
$$-xy + 8$$
$$\overline{\phantom{xxxx}}$$
$$4xy - 16$$

$$2y^2 - y - 1$$
$$y^2 + y + 1$$
$$\overline{\phantom{xxxx}}$$
$$y^2 - 2y - 2$$

$$-10 - 8x - 3xy$$
$$xy - 7x + 3 - 4ay$$
$$\overline{\phantom{xxxx}}$$
$$-13 - x - 4xy + 4ay$$

$$5x^2y - 8$$
$$-3x^2y + 1$$

$$4\sqrt{xy} - x\sqrt{xy}$$
$$2\sqrt{xy} + 2 + xy$$

$$5x^2 + \sqrt{x} - 8 - 4b$$
$$6x^2 - 10 + 4b - x^{\frac{1}{2}}$$

---

* This rule is founded on the conſideration, that addition and ſubtraction are oppoſite to each other in their nature and operation, as are the ſigns + and —, by which they are expreſſed and repreſented. So that, ſince to unite a negative quantity with a poſitive one of the ſame kind, has the effect of diminiſhing it, or ſubducting an equal poſitive one from it, therefore to ſubtract a poſitive (which is the oppoſite of uniting or adding) is to add the equal negative quantity. In like manner, to ſubtract a negative quantity, is the ſame in effect as to add or unite an equal poſitive one. So that, by changing the ſign of a quantity from + to —, or from — to +, changes its nature from a ſubductive quantity to an additive one; and any quantity is in effect ſubtracted, by barely changing its ſign.

$3xy$

$$3xy - 20 \qquad 4x^3 - 3. \ (a + b) \qquad xy^3 + 10a\sqrt{(xy + 10)}$$
$$3xy - 30 \qquad 3x^2 - 8. \ (a + b) \qquad x^2y^2 + 2a\sqrt{(xy + 10)}$$

From $a + b$, take $a - b$.

From $4a + 4b$, take $b + a$.

From $4a - 4b$, take $5b + 3a$.

From $8a - 12x$, take $- 4x + 3a$.

From $2x - 4a - b + 5$, take $8 - 5b + a + 6x$.

From $3a + b + c - d - 10$, take $c + a - d$.

From $3a + b + c - d - 10$, take $b - 10 + 3a$.

From $2ab + b^2 - c + bc - b$, take $3a^2 - c + b^2$.

From $a^3 + b^2c + ab^2 - abc$, take $b^2 + ab^2 - abc$.

From $12x + 6a - 4b + 40$, take $4b - 3a + 2x + 6d - 10$.

From $2x - 3a + 4b + 6c - 50$, take $9a + x + 8b - 6c - 40$.

From $6a - 4b - 12c + 12x$, take $2x - 3a + 4b - 5c$.

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# MULTIPLICATION.

## CASE I.

### *When both the Factors are Simple Quantities.*

### RULE.

MULTIPLY the co-efficients of the two terms together, then to the product affix all the letters in those terms, and the result will be the whole product required.

*Note\**. Like signs produce $+$, and unlike signs $-$.

EXAM-

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\* That like signs make $+$, and unlike signs $-$, in the product, may be shewn thus.

1. When $+ a$ is to be multiplied by $+ b$; this implies that $+ a$ is to be taken as many times as there are units in $b$; and since the sum of any number of affirmative terms is affirmative, it follows that $+ a \times + b$ makes $+ ab$.

2. When

## EXAMPLES.

| | | | |
|---|---|---|---|
| $12a$ | $-2a$ | $5a$ | $-9x$ |
| $3b$ | $+4b$ | $-6x$ | $-5b$ |
| $36ab$ | $-8ab$ | $-30ax$ | $+45bx$ |

| | | | |
|---|---|---|---|
| $7ab$ | $6a^2x$ | $-x^2y$ | $-7xy$ |
| $-5ac$ | $5x$ | $xy^2$ | $-xy$ |
| $-35a^2bc$ | $30a^2x^2$ | $-x^3y^3$ | $+7x^2y^2$ |

| | | | |
|---|---|---|---|
| $-5ax$ | $-ax$ | $+5xy$ | $-7xyz$ |
| $3x$ | $-7b$ | $-3$ | $-6ax$ |

## CASE II.

*When one of the Factors is a Compound Quantity.*

### RULE.

MULTIPLY every term of the multiplicand, separately, by the multiplier, as in the former case; placing the products one after another, with the proper signs; and the result will be the whole product required.

2. When two quantities are to be multiplied together, the result will be exactly the same, in whatever order they are placed; for $a$ times $b$ is the same as $b$ times $a$; and therefore, when $-a$ is to be multiplied by $+b$, or $+b$ by $-a$: this is the same thing as taking $-a$ as many times as there are units in $+b$; and since the sum of any number of negative terms is negative, it follows that $-a \times +b$, or $+a \times -b$ make or produce $-ab$.

3. When $-a$ is to be multiplied by $-b$: here $-a$ is to be subtracted as often as there are units in $b$: but subtracting negatives is the same thing as adding affirmatives, by the demonstration of the rule for subtraction; consequently the product is $b$ times $a$, or $+ab$.

Otherwise. Since $a - a = 0$, therefore $(a-a) \times -b$ is also $= 0$, because 0 multiplied by any quantity, is still but 0; and since the first term of the product, or $a \times -b$ is $= -ab$, by the second case; therefore the last term of the product, or $-a + -b$, must be $+ab$, to make the sum $= 0$, or $-ab \mp ab = 0$; that is, $-a \times -b = +ab$.

EXAM-

### EXAMPLES.

$$4a - 2b$$
$$3a$$
$$\overline{\quad\quad\quad}$$
$$12a^2 - 6ab$$

$$6xy - 8$$
$$2x$$
$$\overline{\quad\quad\quad}$$
$$12x^2y - 16x$$

$$a^2 - 2x + 6$$
$$xy$$
$$\overline{\quad\quad\quad\quad\quad}$$
$$a^2xy - 2x^2y + 6xy$$

$$13x - ab$$
$$12a$$

$$35x - 7a$$
$$-x$$

$$3y - 8 + 2xy$$
$$xy$$

$$2x^2 + x$$
$$2xy$$

$$12x^2 - 4y^2$$
$$-2x^2$$

$$2y^2 - 8x^2 - 7x$$
$$3xy^2$$

### CASE III.

*When both the Factors are Compound Quantities.*

### RULE.

MULTIPLY every term of the multiplier into every term of the multiplicand, respectively; setting down the products one after another, with their proper signs; and add the several lines of products all together for the whole product required.

$$x + y$$
$$x + y$$
$$\overline{\quad\quad\quad}$$
$$x^2 + xy$$
$$\quad + xy + y^2$$
$$\overline{\quad\quad\quad\quad}$$
$$x^2 + 2xy + y^2$$

$$5x + 4y$$
$$3x - 2y$$
$$\overline{\quad\quad\quad}$$
$$15x^2 + 12xy$$
$$\quad - 10xy - 8y^2$$
$$\overline{\quad\quad\quad\quad}$$
$$15x^2 + 2xy - 8y^2$$

$$x^2 + xy - y^2$$
$$x - y$$
$$\overline{\quad\quad\quad\quad}$$
$$x^3 + x^2y - xy^2$$
$$\quad - x^2y - xy^2 + y^3$$
$$\overline{\quad\quad\quad\quad\quad}$$
$$x^3 * - 2xy^2 + y^3$$

$$x + y$$
$$x - y$$
$$\overline{\quad\quad}$$
$$x^2 + xy$$
$$\quad - xy - y^2$$
$$\overline{\quad\quad\quad}$$
$$x^2 * - y^2$$

$$x^2 + y$$
$$x^2 + y$$
$$\overline{\quad\quad}$$
$$x^4 + yx^2$$
$$\quad + yx^2 + y^2$$
$$\overline{\quad\quad\quad}$$
$$x^4 + 2yx^2 + y^2$$

$$x^2 + xy + y^2$$
$$x - y$$
$$\overline{\quad\quad\quad\quad}$$
$$x^3 + x^2y + xy^2$$
$$\quad - x^2y - xy^2 - y^3$$
$$\overline{\quad\quad\quad\quad\quad}$$
$$x^3 * * - y^3$$

*Note.*

*Note.* In the multiplication of compound quantities, it is the beft way to fet them down in order, according to the powers and the letters of the alphabet. And in multiplying them, begin at the left hand fide, and multiply from the left hand towards the right, in the manner that we write, which is contrary to the way of multiplying numbers. But in fetting down the feveral products, as they arife, in the fecond and following lines, range them under the like terms in the lines above, when there are fuch like quantities; which is the eafieft way for adding them up together.

In many cafes, the multiplication of compound quantities is only to be performed by fetting them down one after another, each within or under a vinculum, with a fign of multiplication between them. As $(a + b) \times (a + b) \times 3ab$, or $\overline{a + b} \cdot \overline{a - b} \cdot 3ab$.

### EXAMPLES FOR PRACTICE.

1. Multiply $12ax$ by $3a$.  Anf. $36a^2x$.
2. Multiply $4x^2 - 2y$ by $2y$.  Anf. $8x^2y - 4y^2$.
3. Multiply $2x + 4y$ by $2x - 4y$.  Anf. $4x^2 - 16y^2$.
4. Multiply $x^2 - xy + y^2$ by $x + y$.  Anf. $x^3 + y^3$.
5. Multiply $x^3 + x^2y + xy^2 + y^3$ by $x - y$. Anf. $x^4 - y^4$.
6. Multiply $x^2 + xy + y^2$ by $x^2 - xy + y^2$.
7. Multiply $3x^2 - 2xy + 5$ by $x^2 + 2xy - 3$.
8. Multiply $2a^2 - 3ax + 4x^2$ by $5a^2 - 6ax - 2x^2$.
9. Multiply $3x^3 + 2x^2y^2 + 3y^3$ by $2x^3 - 3x^2y^2 + 5y^3$.
10. Multiply $a^2 + ab + b^2$ by $a - b$.

## DIVISION.

DIVISION in algebra, like that in numbers, is the converfe of multiplication; and it is performed like that of numbers alfo, by beginning at the left hand fide, and dividing all the parts of the dividend by the divifor, when they can be fo divided; or elfe by fetting them down like a fraction, the dividend over the divifor, and then abbreviating the fraction as much as can be done. This will naturally divide into the following particular cafes.

CASE

## CASE I.

### *When the Divisor and Dividend are both Simple Quantities.*

#### RULE.

SET the terms both down as in division of numbers, either the divisor before the dividend, or below it, like the denominator of a fraction. Then abbreviate thefe terms as much as can be done, by cancelling or ftriking out all the letters that are common to both of them, and alfo dividing the one co-efficient by the other, or abbreviating them after the manner of a fraction, by dividing them by their common meafure.

*Note.* Like figns in the two factors make $+$ in the quotient; and unlike figns make $-$; the fame as in multiplication*.

### EXAMPLES.

1. To divide $8ab$ by $2a$.

Here $8ab \div 2a$, or $2a\,)\,8ab$, or $\dfrac{8ab}{2a} = 4b$.

2. Alfo $a \div a = \dfrac{a}{a} = 1$; and $abc \div bcd = \dfrac{abc}{bcd} = \dfrac{a}{d}$.

3. Divide $16x^2$ by $8x$.      Anf. $2x$.

4. Divide $12a^2 x^2$ by $-3a^2 x$.      Anf. $-4x$.

5. Divide $-15ay^2$ by $3ay$.      Anf. $-5y$.

6. Divide $-18ax^2y$ by $8axz$.      Anf. $\dfrac{9}{4}\dfrac{xy}{z}$.

---

* Becaufe the divifor multiplied by the quotient, muft produce the dividend. Therefore,

1. When both the terms are $+$, the quotient muft be $+$; becaufe $+$ in the divifor $\times$ $+$ in the quotient, produces $+$ in the dividend.

2. When the terms are both $-$. the quotient is alfo $+$; becaufe $-$ in the divifor $\times$ $+$ in the quotient, produces $-$ in the dividend.

3. When one term is $+$ and the other $-$, the quotient muft be $-$; becaufe $+$ in the divifor $\times$ $-$ in the quotient produces $-$ in the dividend, or $-$ in the divifor $\times$ $+$ in the quotient gives $-$ in the dividend.

So that the rule is general, like figns give $+$, and unlike figns give $-$, in the quotient.

CASE

## CASE II.

*When the Dividend is a Compound Quantity, and the Divisor a Simple one.*

### RULE.

DIVIDE every term of the dividend by the divisor, as in the former case.

### EXAMPLES.

1. $(ab + b^2) \div 2b$, or $\dfrac{ab + b^2}{2b} = \dfrac{a + b}{2} = \tfrac{1}{2}a + \tfrac{1}{2}b.$

2. $(10ab + 15ax) \div 5a$, or $\dfrac{10ab + 15ax}{5a} = 2b + 3x.$

3. $(30az - 48z) \div z$, or $\dfrac{30az - 48z}{z} = 30a - 48.$

4. Divide $4ab - 8ax + a$ by $2a$.

5. Divide $3x^2 - 15 + 6x + 3a$ by $3x$.

6. Divide $3abc + 12abx - 9a^2b$ by $3ab$.

7. Divide $10a^2x - 15x^2 - 5x$ by $5x$.

8. Divide $15a^2bc - 12acx^2 + 5ad^2$ by $- 5ac$.

9. Divide $12a + 3ay - 18y^2$ by $21a$.

10. Divide $- 40a^2b^2 + 60ab^3$ by $- 6ab$.

## CASE III.

*When the Divisor and Dividend are both Compound Quantities.*

### RULE.

1. SET them down as in common division of numbers, the divisor before the dividend, with a small crooked line between them, and ranging the terms according to the powers of some one of the letters in both, the higher powers of it before the lower.

2. Divide the first term of the dividend by the first term of the divisor, as in the first case, and place the result in the quotient.

3. Multiply the whole divisor by the term thus found, and subtract the result from the dividend.

4. To this remainder bring down as many terms of the dividend as are requisite for the next operation, dividing as before; and so on to the end, as in common arithmetic.

*Note*

*Note.* If the divifor be not exactly contained in the dividend, the quantity which remains after the operation is finifhed. may be placed over the divifor, like a vulgar fraction, and let down at the end of the quotient, as in common arithmetic.

### EXAMPLES.

$$x + y ) \; x^2 + 2xy + y^2 \; ( x + y$$
$$\phantom{x + y ) \;} x^2 + xy$$

$$\phantom{x + y ) \;\;\;\;\;\;\;\;} xy + y^2$$
$$\phantom{x + y ) \;\;\;\;\;\;\;\;} xy + y^2$$

$$a + x ) \; a^3 + 5a^2 x + 5ax^2 + x^3 \; ( a^2 + 4ax + x^2$$
$$\phantom{a + x ) \;} a^3 + a^2 x$$

$$\phantom{a + x )} 4a^2 x + 5ax^2$$
$$\phantom{a + x )} 4a^2 x + 4ax^2$$

$$\phantom{a + x )\;\;\;\;\;} ax^2 + x^3$$
$$\phantom{a + x )\;\;\;\;\;} ax^2 + x^3$$

$$x - 3 ) \; x^3 - 9x^2 + 27x - 27 \; ( x^2 - 6x + 9$$
$$\phantom{x - 3 ) \;} x^3 - 3x^2$$

$$\phantom{x - 3 )} - 6x^2 + 27x$$
$$\phantom{x - 3 )} - 6x^2 + 18x$$

$$\phantom{x - 3 )\;\;\;\;\;} 9x - 27$$
$$\phantom{x - 3 )\;\;\;\;\;} 9x - 27$$

$$a - x ) \; a^3 - x^3 \; ( a^2 + ax + x^2$$
$$\phantom{a - x ) \;} a^3 - a^2 x$$

$$\phantom{a - x )\;\;} a^2 x - x^3$$
$$\phantom{a - x )\;\;} a^2 x - ax^2$$

$$\phantom{a - x )\;\;\;\;\;} ax^2 - x^3$$
$$\phantom{a - x )\;\;\;\;\;} ax^2 - x^3$$

$$b - y ) \; b^4 - 3y^4 \; ( \; b^3 + b^2y + by^2 + y^3 - \dfrac{2y^4}{b-y}.$$

$$\underline{b^4 - b^3y}$$

$$b^3y - 3y^4$$
$$\underline{b^3y - b^2y^2}$$

$$b^2y^2 - 3y^4$$
$$\underline{b^2y^2 - by^3}$$

$$by^3 - 3y^4$$
$$\underline{by^3 - \; y^4}$$

$$\underline{- 2y^4}$$

### EXAMPLES FOR PRACTICE.

1. Divide $a^2 + 2ax + x^2$ by $a + x$.  Anf. $a + x$.
2. Divide $a^3 - 3a^2y + 3ay^2 - y^3$ by $a - y$.

  Anf. $a^2 - 2ay + y^2$.

3. Divide 1 by $1 - x$.  Anf. $1 + x + x^2 + x^3 + $ &c.
4. Divide $6x^4 - 96$ by $3x - 6$.

  Anf. $2x^3 + 4x^2 + 8x + 16$.

5. Divide $a^5 - 5a^4x + 10a^3x^2 - 10a^2x^3 + 5ax^4 - x^5$
  by $a^2 - 2ax + x^2$.  Anf. $a^3 - 3a^2x + 3ax^2 - x^3$.
6. Divide $48x^3 - 76ax^2 - 64a^2x + 105a^3$ by $2x - 3a$.
7. Divide $y^6 - 3y^4x^2 + 3y^2x^4 - x^6$ by $y^3 - 3y^2x + 3yx^2 - x^3$.
8. Divide $a^5 - x^5$ by $a - x$.
9. Divide $a^3 + 5a^2x + 5ax^2 + x^3$ by $a + x$.
10. Divide $a^4 + 4a^2b^2 + 3b^4$ by $a + 2b$.

## ALGEBRAIC FRACTIONS.

ALGEBRAIC FRACTIONS have the fame names and rules of operation, as numeral fractions in common arithmetic; as appears in the following Problems:

## PROBLEM I.

*To Reduce a Mixed Quantity to an Improper Fraction.*

### RULE.

MULTIPLY the integer by the denominator of the fraction, and to the product add the numerator; then the denominator being placed under this fum, will give the improper fraction required.

### EXAMPLES.

1. Reduce $3\frac{5}{7}$, and $a - \dfrac{b}{c}$ to improper fractions.

Firſt, $3\frac{5}{7} = \dfrac{3 \times 7 + 5}{7} = \dfrac{21 + 5}{7} = \dfrac{26}{7}$ the Anſ.

And, $a - \dfrac{b}{c} = \dfrac{a \times c - b}{c} = \dfrac{ac - b}{c}$ the Anſ.

2. Reduce $x + \dfrac{x^2}{a}$ and $x - \dfrac{a^2 - x^2}{x}$ to improper fractions,

Firſt, $x + \dfrac{x^2}{a} = \dfrac{x \times a + x^2}{a} = \dfrac{ax + x^2}{a}$ the Anſ.

And, $x - \dfrac{a^2 - x^2}{x} = \dfrac{x^2 - a^2 + x^2}{x} = \dfrac{2x^2 - a^2}{x}$ Anſ.

3. Reduce $8\frac{6}{7}$ to an improper fraction. Anſ. $\frac{62}{7}$.

4. Reduce $1 - \dfrac{2x}{a}$ to an improper fraction. Anſ. $\dfrac{a - 2x}{a}$.

5. Reduce $x - \dfrac{ax + x^2}{2a}$ to an improper fraction.

6. Reduce $10 + \dfrac{2x - 8}{3x}$ to an improper fraction.

7. Reduce $a + \dfrac{1 - x - b}{b}$ to an improper fraction.

8. Reduce $1 + 2x - \dfrac{x - 3}{5x}$ to an improper fraction.

## PROBLEM II.

*To reduce an Improper Fraction to a Whole or Mixed Quantity.*

### RULE.

DIVIDE the numerator by the denominator, for the integral part; and place the remainder, if any, over the denominator, for the fractional part; the two joined together will be the mixed quantity required.

O 2

### EXAMPLES.

1. To reduce $\frac{17}{5}$, and $\dfrac{ax + a^2}{x}$ to mixed quantities.

Firſt, $\frac{17}{5} = 17 \div 5 = 3\frac{2}{5}$, the Anſwer required.

And $\dfrac{ax + a^2}{x} = \overline{ax + a^2} \div x = a + \dfrac{a^2}{x}$ Anſ.

2. To reduce $\dfrac{ab - a^2}{b}$, and $\dfrac{ay + 2y^2}{a + y}$ to mixed quantities.

Firſt, $\dfrac{ab - a^2}{b} = \overline{ab - a^2} \div b = a - \dfrac{a^2}{b}$ Anſ.

And, $\dfrac{ay + 2y^2}{a + y} = \overline{ay + 2y^2} \div \overline{a + y} = y + \dfrac{y^2}{a + y}$.

3. Let $\frac{35}{8}$, and $\dfrac{3ab - b^2}{a}$ be reduced to whole or mixed quantities.      Anſ. $4\frac{3}{8}$, and $3b - \dfrac{b^2}{a}$.

4. Let $\dfrac{2x^2 y}{2x}$, and $\dfrac{a^2 + x^2}{a - x}$ be reduced to whole or mixed quantities.

5. Let $\dfrac{x^2 - y^2}{x + y}$, and $\dfrac{x^3 - y^3}{x - y}$ be reduced to whole or mixed quantities.

6. Reduce $\dfrac{10x^2 - x + 3}{5x}$ to a mixed quantity.

7. Reduce $\dfrac{12x^3 + 3x^2}{4x^3 + x^2 - 4x - 1}$ to a mixed quantity.

### PROBLEM III.

*To Reduce Fractions to a Common Denominator.*

#### RULE.

MULTIPLY every numerator, ſeparately, into all the denominators except its own, for the new numerators ; and all the denominators together, for the common denominator*.

---

* When the denominators have a common diviſor, it will be better, inſtead of multiplying by the whole denominators, to multiply only by thoſe parts which ariſe from dividing by the common diviſor.

## EXAMPLES.

1. Reduce $\frac{a}{b}$ and $\frac{b}{c}$ to a common denominator.

$$\left.\begin{array}{l} a \times c = ac \\ b \times b = b^2 \end{array}\right\} \text{the new numerators.}$$

$b \times c = bc$ the common denominator.

therefore $\frac{a}{b}$ and $\frac{b}{c} = \frac{ac}{bc}$ and $\frac{b^2}{bc}$ the fractions required.

2. Reduce $\frac{a}{b}$, $\frac{b}{c}$, and $\frac{c}{d}$ to a common denominator.

$$\left.\begin{array}{l} a \times c \times d = acd \\ b \times b \times d = b^2 d \\ c \times b \times c = c^2 b \end{array}\right\} \text{the numerators.}$$

$b \times c \times d = bcd$ the com. denom.

therefore $\frac{a}{b}$, $\frac{b}{c}$, and $\frac{c}{d} = \frac{acd}{bcd}$, $\frac{b^2 d}{bcd}$ and $\frac{c^2 b}{bcd}$, the frac. required.

3. Reduce $\frac{2x}{a}$ and $\frac{b}{c}$ to equivalent fractions, having a common denominator.   Anf. $\frac{2cx}{ac}$ and $\frac{ab}{ac}$.

4. Reduce $\frac{a}{b}$ and $\frac{a+b}{c}$ to fractions having a common denominator.   Anf. $\frac{ac}{bc}$ and $\frac{ab+b^2}{bc}$.

5. Reduce $\frac{3x}{2a}$, $\frac{2b}{3c}$, and $d$, to fractions having a common denominator.   Anf. $\frac{9cx}{6ac}$, $\frac{4ab}{6ac}$, and $\frac{6acd}{6ac}$.

6. Reduce $\frac{3}{4}$, $\frac{2x}{3}$ and $a + \frac{2x}{a}$, to fractions having a common denominator.   Anf. $\frac{9a}{12a}$, $\frac{8ax}{12a}$, and $\frac{12a^2 + 24x}{12a}$.

7. Reduce $\frac{1}{2}$, $\frac{a^2}{3}$ and $\frac{a^2 + x^2}{a+x}$ to a common denominator.

8. Reduce $\frac{b}{2a^2}$, $\frac{c}{2a}$, and $\frac{d}{a}$ to a common denominator.

I         PRO-

## PROBLEM IV.

*To find the Greatest Common Measure of the Terms of a Fraction.*

### RULE*.

1. RANGE the quantities according to the dimensions of some letters, as is shown in division.

2. Divide the greater term by the less, and the last divisor by the last remainder, and so on till nothing remains; then the divisor last used will be the common measure required.

*Note.* All the letters or figures which are common to each term of the divisors, must be thrown out of them, before they are used in the operation.

### EXAMPLES.

1. To find the greatest common measure of $\dfrac{cx + x^2}{ca^2 + a^2x}$.

$$cx + x^2 \;)\; ca^2 + a^2x$$
$$\text{or } \quad c + x \;)\; ca^2 + a^2x \;(\; a^2$$
$$ca^2 + a^2x$$

Therefore the greatest common measure is $c + x$.

2. To find the greatest common measure of $\dfrac{x^3 - b^2x}{x^2 + 2bx + b^2}$.

$$x^2 + 2bx + b^2 \;)\; x^3 - \quad b^2x \;(\; x$$
$$x^3 + 2bx^2 + b^2x$$

$$- 2bx^2 - 2b^2x \;(\; x^2 + 2bx + b^2$$
$$\text{or} \quad x + b \quad (\; x^2 + 2bx + b^2 \;(\; x + b$$
$$x^2 + \quad bx$$

$$bx + b^2$$
$$bx + b^2$$

Therefore $x + b$ is the greatest common divisor.

3. To find the greatest common divisor of $\dfrac{x^2 - 1}{xy + y}$.

Ans. $x + 1$.

---

* The simple divisors, in this rule, may be easily found, by inspection.

4. To

4. To find the greateſt common diviſor of $\dfrac{x^4 - b^4}{x^5 + b^2 x^3}$.

Anſ. $x^2 + b^2$.

5. To find the greateſt common meaſure of

$$\dfrac{5a^5 + 10a^4 b + 5a^3 b^2}{a^3 b + 2a^2 b^2 + 2ab^3 + b^4}.$$

## PROBLEM V.

*To Reduce a Fraction to its Loweſt Terms.*

### RULE.

1. FIND the greateſt common meaſure, as in the laſt problem.

2. Divide both the terms of the fraction by the common meaſure thus found, and it will reduce it to its loweſt terms, as was required.

3. Or divide the terms by any quantity which it may appear will divide them both.

### EXAMPLES.

1. Reduce $\dfrac{cx + x^2}{ca^2 + a^2 x}$ to its loweſt terms.

$$cx + x^2 \;)\; ca^2 + a^2 x$$
$$\text{or } c + x \quad)\; ca^2 + a^2 x \;(\; a^2$$
$$\qquad\qquad ca^2 + a^2 x$$
$$\qquad\qquad \overline{\hspace{3cm}}$$

Here $cx + x^2$ is divided by $x$ which is common to both terms.

Therefore $c + x$ is the greateſt common meaſure,

and $c + x \;)\; \dfrac{cx + x^2}{ca^2 + a^2 x} = \dfrac{x}{a^2}$, is the fraction required.

3. Having $\dfrac{x^3 - b^2 x}{x^2 + 2bx + b^2}$ given, it is required to reduce it to its leaſt terms.

$$x^2 + 2bx + b^2 \;)\; x^3 - b^2 x \;(\; x$$
$$\qquad\qquad x^3 + 2bx^2 + b^2 x$$
$$\qquad\qquad \overline{\hspace{4cm}}$$
$$-2bx^2 - 2b^2 x \;)\; x^2 + 2bx + b^2$$
$$\text{or } x + b \;)\; x^2 + 2bx + b^2 \;(\; x + b$$
$$\qquad\qquad x^2 + bx$$
$$\qquad\qquad \overline{\hspace{3cm}}$$
$$\qquad\qquad\qquad bx + b^2$$
$$\qquad\qquad\qquad bx + b^2$$
$$\qquad\qquad\qquad \overline{\hspace{2cm}}$$

Therefore

Therefore $x + b$ is the greateſt common meaſure,

and $x + b$)$\dfrac{x^3 - b^2x}{x^2 + 2bx + b^2} = \dfrac{x^2 - bx}{x + b}$ is the fraction required.

3. Reduce $\dfrac{x^4 - b^4}{x^5 - b^2x^3}$ to its loweſt terms. Anſ. $\dfrac{x^2 + b^2}{x^3}$.

4. Reduce $\dfrac{x^2 - y^2}{x^4 - y^4}$ to its loweſt terms. Anſ. $\dfrac{1}{x^2 + y^2}$.

5. Reduce $\dfrac{a^4 - x^4}{a^3 - a^2x - ax^2 + x^3}$ to its loweſt terms.

6. Reduce $\dfrac{5a^5 + 10a^4x + 5a^3x^2}{a^3x + 2a^2x^2 + 2ax^3 + x^4}$ to its loweſt terms.

## PROBLEM VI.

*To Add Fractional Quantities together.*

### RULE.

1. If the fractions have a common denominator, add all the numerators together, then under their ſum write the common denominator, and it will give the ſum of the fractions required.

2. If they have not a common denominator, reduce them to a common denominator, and then add them as before.

### EXAMPLES.

1. Having $\dfrac{x}{2}$ and $\dfrac{x}{3}$ given, to find their ſum.

Here $\left.\begin{array}{l} x \times 3 = 3x \\ x \times 2 = 2x \end{array}\right\}$ the numerators.

and $2 \times 3 = 6$ the com. denom.

theref. $\dfrac{3x}{6} + \dfrac{2x}{6} = \dfrac{5x}{6}$ is the ſum required.

2. Having $\dfrac{a}{b}$, $\dfrac{c}{d}$, and $\dfrac{e}{f}$ given, to find their ſum.

Here $\left.\begin{array}{l} a \times d \times f = adf \\ c \times b \times f = cbf \\ e \times b \times d = ebd \end{array}\right\}$ the numerators.

and $b \times d \times f = bdf$ the com. denom.

theref. $\dfrac{adf}{bdf} + \dfrac{cbf}{bdf} + \dfrac{ebd}{bdf} = \dfrac{adf + cbf + ebd}{bdf}$ the ſum required.

3. Let

*3. Let $a - \dfrac{3x^2}{b}$ and $b + \dfrac{2ax}{c}$ be added together.

$$\left.\begin{array}{l} 3x^2 \times c = 3cx^2 \\ 2ax \times b = 2abx \end{array}\right\} \text{the numerators.}$$

and $b \times c = bc$ the common denominator.

Therefore $a - \dfrac{3x^2}{b} + b + \dfrac{2ax}{c} = a - \dfrac{3cx^2}{bc} + b + \dfrac{2abx}{bc} =$

$a + b + \dfrac{2abx - 3cx^2}{bc}$ the sum required.

4. Add $\dfrac{3x}{2b}$ and $\dfrac{x}{5}$ together.     Anf. $\dfrac{15x + 2bx}{10b}$.

5. Add $\dfrac{x}{2}$, $\dfrac{x}{3}$ and $\dfrac{x}{4}$ together.   Anf. $\dfrac{13}{12}x$ or $x + \dfrac{x}{12}$.

6. Add $\dfrac{x - 2}{3}$ and $\dfrac{4x}{7}$ together.     Anf. $\dfrac{19x - 14}{21}$.

7. Add $x + \dfrac{x - 2}{3}$ to $3x + \dfrac{2x - 3}{4}$. Anf. $4x + \dfrac{10x - 17}{12}$.

8. It is required to add $4x$, $\dfrac{5x^2}{2a}$ and $\dfrac{x + a}{2x}$ together.

9. It is required to add $\dfrac{2x}{3}$, $\dfrac{7x}{4}$ and $\dfrac{2x + 1}{5}$ together.

10. It is required to add $4x$, $\dfrac{7x}{9}$ and $2 + \dfrac{x}{5}$ together.

11. It is required to add $3x + \dfrac{2x}{5}$ and $x - \dfrac{8x}{9}$ together.

## PROBLEM VII.

*To Subtract one Fractional Quantity from another.*

### RULE†.

1. REDUCE the fractions to a common denominator, as in addition, if they have not a common denominator.

2. Subtract the numerators from each other, and under their difference write the common denominator, and it will give the difference of the fractions required.

---

* In the addition of mixed quantities, it is beft to bring the fractional parts only to a common denominator, and to affix their fum to the fum of the integers, interpofing the proper fign.

† The fame rule may be obferved for mixed quantities, in fubtraction, as in addition.

## EXAMPLES.

1. To find the difference of $\dfrac{x}{3}$ and $\dfrac{2x}{11}$.

Here $\left.\begin{array}{c} x \times 11 = 11x \\ 2x \times 3 = 6x \end{array}\right\}$ the numerators.

and $\quad 3 \times 11 = 33$ the common denominator.

theref. $\dfrac{11x}{33} - \dfrac{6x}{33} = \dfrac{5x}{33}$ is the difference required.

2. To find the difference of $\dfrac{x - a}{3b}$ and $\dfrac{2a - 4x}{5c}$.

Here $\left.\begin{array}{c} x - a \times 5c = 5cx - 5ac \\ 2a - 4x \times 3b = 6ab - 12bx \end{array}\right\}$ the nu-merators.

and $\quad 3b \times 5c = 15bc$ the com. denom.

Then $\dfrac{5cx - 5ac}{15bc} - \dfrac{6ab - 12bx}{15bc} = \dfrac{5cx - 5ac - 6ab + 12bx}{15bc}$

is the difference required.

3. Required the difference of $\dfrac{12x}{7}$ and $\dfrac{3x}{5}$.

4. Required the difference of $5y$ and $\dfrac{3y}{8}$.

5. Required the difference of $\dfrac{3x}{7}$ and $\dfrac{2x}{9}$.

6. Subtract $\dfrac{c}{d}$ from $\dfrac{x + a}{b}$.

7. Take $\dfrac{2x + 7}{8}$ from $\dfrac{3x + a}{5b}$.

8. Take $x - \dfrac{x - a}{c}$ from $3x + \dfrac{x}{b}$.

### PROBLEM VIII.

*To Multiply Fractional Quantities together.*

#### RULE *.

MULTIPLY the numerators together, for a new nume-rator, and the denominators for a new denominator; and it will give the product required.

EXAM-

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* 1. When the numerator of one fraction, and the denominator of the other, can be divided by some quantity, which is common to both. the quotients may be used instead of them.

2. When a fraction is to be multiplied by an integer, the product is found by multiplying the numerator by it; and if the integer be the

## EXAMPLES.

1. Required to find the product of $\frac{x}{6}$ and $\frac{2x}{9}$.

Here $\frac{x \times 2x}{6 \times 9} = \frac{2x^2}{54} = \frac{x^2}{27}$ the product required.

2. Required the product of $\frac{x}{2}$, $\frac{4x}{5}$, and $\frac{10x}{21}$.

$\frac{x \times 4x \times 10x}{2 \times 5 \times 21} = \frac{40x^3}{210} = \frac{4x^3}{21}$ the product required.

3. Required the product of $\frac{x}{a}$ and $\frac{x+a}{a+c}$.

Here $\frac{x \times (x+a)}{a \times (a+c)} = \frac{x^2+ax}{a^2+ac}$ the product required.

4. Required the product of $\frac{3x}{2}$ and $\frac{2a}{b}$.

5. Required the product of $\frac{2x}{5}$ and $\frac{3x^2}{2a}$.

6. To multiply $\frac{2x}{a}$, and $\frac{3ab}{c}$, and $\frac{3ac}{2b}$ together.

7. Required the product of $b + \frac{bx}{a}$ and $\frac{a}{x}$.

8. Required the product of $\frac{x^2 - b^2}{bc}$ and $\frac{x^2 + b^2}{b + c}$.

9. Required the product of $x$, and $\frac{x+1}{a}$, and $\frac{x-1}{a+b}$.

10. Multiply $a + \frac{x}{2a} - \frac{x^2}{4a^2}$ by $x - \frac{a}{2x} + \frac{a^2}{4x^2}$.

## PROBLEM IX.

*To Divide one Fractional Quantity by another.*

### RULE*.

MULTIPLY the denominator of the divisor by the numerator of the dividend, for a new numerator, and the numerator of the divisor by the denominator of the dividend, for a new denominator.

Or,

---

the same with the denominator, the numerator may be taken for the product.

3. When a fraction is to be multiplied by any quantity, it is the same thing whether the numerator be multiplied by it, or the denominator divided by it.

* 1. If the fractions to be divided have a common denominator, take the numerator of the dividend for a new numerator, and the numerator of the divisor for the denominator.

2. When

Or, invert the terms of the divifor, and then multiply by it exactly as in multiplication.

## EXAMPLES.

1. Required the quotient of $\dfrac{x}{3}$ divided by $\dfrac{2x}{9}$.

Here $\dfrac{x}{3} \times \dfrac{9}{2x} = \dfrac{9x}{6x} = \dfrac{3}{2} = 1\frac{1}{2}$ is the quot. required.

2. Required the quotient of $\dfrac{2a}{b}$ divided by $\dfrac{4c}{d}$.

Here $\dfrac{2a}{b} \times \dfrac{d}{4c} = \dfrac{2ad}{4bc} = \dfrac{ad}{2bc}$ is the quotient required.

3. Find the quotient of $\dfrac{x+a}{2x-2b}$ divided by $\dfrac{x+b}{5x+a}$.

$\dfrac{x+a}{2x-2b} \times \dfrac{5x+a}{x+b} = \dfrac{5x^2 + 6ax + a^2}{2x^2 - 2b^2}$ the quot. required.

4. Find the quotient of $\dfrac{2x^2}{a^3+x^3}$ divided by $\dfrac{x}{x+a}$.

$\dfrac{2x^2}{a^3+x^3} \times \dfrac{x+a}{x} = \dfrac{2x^2 \times (x+a)}{(a^3+x^3) \times x} = \dfrac{2x}{x^2 - ax + a^2}$ is the quotient required.

5. Let $\dfrac{7x}{5}$ be divided by $\frac{1}{4}\frac{2}{3}$.

6. Let $\dfrac{4x^2}{7}$ be divided by $5x$.

7. Let $\dfrac{x+1}{6}$ be divided by $\dfrac{2x}{3}$.

8. Let $\dfrac{x}{x-1}$ be divided by $\dfrac{x}{2}$.

9. Let $\dfrac{5x}{3}$ be divided by $\dfrac{2a}{3b}$.

10. Let $\dfrac{x-b}{8cd}$ be divided by $\dfrac{3cx}{4d}$.

11. Divide $\dfrac{x^4 - b^4}{x^2 - 2bx + b^2}$ by $\dfrac{x^2 + bx}{x-b}$.

---

2. When a fraction is to be divided by any quantity, it is the fame thing whether the numerator be divided by it, or the denominator multiplied by it.

3. When the two numerators, or the two denominators, can be divided by fome common quantity, that quantity may be thrown out of each, and the quotients ufed inftead of the fractions firft propofed.

INVO-

# INVOLUTION.

INVOLUTION is the raifing of powers from any propofed root; or the method of finding the fquare, cube, biquadrate, &c, of any given quantity.

## RULE*.

MULTIPLY the quantity into itfelf as many times as there are units in the index lefs one, and the laft product will be the power required. Or,

Multiply the index of the quantity by the index of the power, and the refult will be the fame as before.

*Note.* When the fign of the root is +, all the powers of it will be +; and when the fign is —, all the even powers will be +, and all the odd powers —, as is evident from mulplication.

## EXAMPLES.

$a$, root
$a^2$ = fquare
$a^3$ = cube
$a^4$ = 4th power
$a^5$ = 5th power
&c.

$a^2$, root
$a^4$ = fquare
$a^6$ = cube
$a^8$ = 4th power
$a^{10}$ = 5th power
&c.

---

— $3a$, root
+ $9a^2$ = fquare
— $27a^3$ = cube
+ $81a^4$ = 4th power
— $243a^5$ = 5th power

— $2ax^2$, root
+ $4a^2x^4$ = fquare
— $8a^3x^6$ = cube
+ $16a^4x^8$ = 4th power
— $32a^5x^{10}$ = 5th power

---

— $\dfrac{2ax^2}{3b}$, root

+ $\dfrac{4a^2x^4}{9b^2}$ = fquare

— $\dfrac{8a^3x^6}{27b^3}$ = cube

+ $\dfrac{16a^4x^8}{81b^4}$ = 4th power

$\dfrac{x}{a}$, root

$\dfrac{x^2}{a^2}$ = fquare

$\dfrac{x^3}{a^3}$ = cube

$\dfrac{x^4}{a^4}$ = biquadrate

---

* Any power of the product of two or more quantities is equal to the fame power of each of the factors, multiplied together.

And any power of a fraction is equal to the fame power of the numerator, divided by the like power of the denominator.

Alfo, powers or roots of the fame quantity, are multiplied by one another, by adding their exponents; or divided, by fubtracting their exponents.

Thus, $a^3 \times a^2 = a^{3+2} = a^5$. And $a^3 \div a^2$ or $\dfrac{a^3}{a^2} = a^{3-2} = a$.

$x — a.$

$x - a =$ root

$x - a$

---

$x^2 - ax$

$\quad - ax + a^2$

---

$x^2 - 2ax + a^2$ fquare

$x \quad - a$

---

$x^3 - 2ax^2 + a^2 x$

$\quad - ax^2 + 2a^2 x - a^3$

---

$x^3 - 3ax^2 + 3a^2 x - a^3$

$x + a =$ root

$x + a$

---

$x^2 + ax$

$\quad + ax + a^2$

---

$x^2 + 2ax + a^2$

$x \quad + a$

---

$x^3 + 2ax^2 + a^2 x$

$\quad + ax^2 + 2a^2 x + a^3$

---

$x^3 + 3ax^2 + 3a^2 x + a^3$

the cubes, or third powers of $x + a$ and $x - a$.

### EXAMPLES FOR PRACTICE.

1. Required the cube or 3d power of $2a^2$.

2. Required the 4th power of $2a^2 x$.

3. Required the 3d power of $-8x^2 y^3$.

4. To find the biquadrate of $-\dfrac{2a^2 x}{3b^2}$.

5. To find the 6th power of $a^{\frac{1}{2}}$.

6. Required the 5th power of $a - x$.

Sir ISAAC NEWTON'S RULE *for raifing a Binomial or Refidual Quantity to any Power whatever*[*].

1. *To find the Terms without the Co-efficients.* The index of the firft, or leading quantity, begins with that of the given power, and decreafes continually by 1, in every term to the laft; and in the following quantity, the indices of the terms are 0, 1, 2, 3, 4, &c.

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* This rule, expreffed in general terms, is as follows:

$$\overline{a + b}^n = a^n + n.a^{n-1}b + n.\frac{n-1}{2} a^{n-2}b^2 + n.\frac{n-1}{2}.\frac{n-2}{3} a^{n-3} b^3 \ \&c.$$

$$\overline{a - b}^n = a^n - n.a^{n-1}b + n.\frac{n-1}{2} a^{n-2}b^2 - n.\frac{n-1}{2}.\frac{n-2}{3} a^{n-3} b^3 \ \&c.$$

*Note.* The fum of the co-efficients, in every power, is equal to the number 2, raifed to that power. Thus $1 + 1 = 2$ for the firft power; $1 + 2 + 1 = 4 = 2^2$ for the fquare; $1 + 3 + 3 + 1 = 8 = 2^3$ for the cube, or third power; and fo on

2. To

2. *To find the Unciæ, or Co-efficients.* The firſt is always 1, and the ſecond is the index of the power; and in general, if the co-efficient of any term be multiplied by the index of the leading quantity, and the product be divided by the number of terms to that place, it will give the co-efficient of the term next following.

*Note.* The whole number of terms will be one more than the index of the given power; and when both terms of the root are +, all the terms of the power will be +; but if the ſecond term be —, all the old terms will be +, and all the even terms —, which cauſes the terms to be + and — alternately.

### E X A M P L E S.

1. Let $a + x$ be involved to the 5th power.

The terms without the co-efficients will be
$$a^5, \quad a^4x, \quad a^3x^2, \quad a^2x^3, \quad ax^4, \quad x^5,$$
and the co-efficients will be
$$1, 5, \frac{5 \times 4}{2}, \quad \frac{10 \times 3}{3}, \quad \frac{10 \times 2}{4}, \quad \frac{5 \times 1}{5};$$
or 1, 5, 10, 10, 5, 1;
And therefore the 5th power altogether is
$$a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5.$$

2. Let $x - a$ be involved to the 6th power.
The terms without the co-efficients will be
$$x^6, \quad x^5a, \quad x^4a^2, \quad x^3a^3, \quad x^2a^4, \quad xa^5, \quad a^6.$$
and the co-efficients will be
$$1, 6, \frac{6 \times 5}{2}, \quad \frac{15 \times 4}{3}, \quad \frac{20 \times 3}{4}, \quad \frac{15 \times 2}{5}, \quad \frac{6 \times 1}{6},$$
or 1, 6, 15, 20, 15, 6, 1.
And therefore the 6th power of $x - a$ is
$$x^6 - 6x^5a + 15x^4a^2 - 20x^3a^3 + 15x^2a^4 - 6xa^5 + a^6.$$

3. Required the 4th power of $x - a$.
$$\text{Anſ. } x^4 - 4x^3a + 6x^2a^2 - 4xa^3 + a^4.$$

4. Required the 7th power of $x + a$.
$$\text{Anſ. } x^7 + 7x^6a + 21x^5a^2 + 35x^4a^3 + 35x^3a^4 + 21x^2a^5 + 7xa^6 + a^7.$$

EVO-

# EVOLUTION.

EVOLUTION is the reverse of Involution, being the method of finding the square root, cube root, &c, of any given quantity, whether simple or compound.

## CASE I.

*To find the Roots of Simple Quantities.*

### RULE*.

EXTRACT the root of the co-efficient, for the numeral part; and divide the index of the letter, or letters, by the index of the power, and it will give the root of the literal part; then annex this to the former, for the whole root fought.

### EXAMPLES.

1. Required the square root of $9x^2$; and the cube root of $8x^3$.

Anf. $\sqrt{9x^2} = 3x^{\frac{2}{2}} = 3x$.  And $\sqrt[3]{8x^3} = 2x^{\frac{3}{3}} = 2x$.

2. To find the fq. root of $\dfrac{3x^2y^2}{4a^2}$, and cube root of $\dfrac{16x^4y^3}{27a^3}$:

Anf. $\sqrt{\dfrac{3x^2y^2}{4a^2}} = \dfrac{xy}{2a}\sqrt{3}$.  And $\sqrt[3]{\dfrac{16x^4y^3}{27a^3}} = \dfrac{2xy}{3a}\sqrt[3]{2x}$.

---

\* Any even root of an affirmative quantity, may be either $+$ or $-$ : thus the fquare root of $+ a^2$ is either $+ a$, or $- a$ ; for $+ a \times + a = + a^2$, and $- a \times - a = + a^2$ alfo.

And an odd root of any quantity will have the fame fign as the quantity itfelf : thus the cube root of $\times a^3$ is $+ a$, and the cube root of $- a^3$ is $- a$; for $+ a \times + a \times + a = + a^3$, and $- a \times - a \times - a = - a^3$.

Any even root of a negative quantity is impoffible; for neither $+ a \times + a$, nor $- a \times - a$ can produce $- a^2$.

Any root of a product, is equal to the like root of each of the factors multiplied together.

And any root of a fraction, is equal to the like root of the numerator, divided by the fame root of the denominator.

3. Re-

3. Required the fquare root of $3a^2x^6$.     Anf. $ax^3 \sqrt{3}$.
4. Required the cube root of $- 125a^3x^6$.   Anf. $- 5ax^2$.
5. Required the fquare root of $\dfrac{9x^2y^2}{4a^3}$.     Anf. $\dfrac{3xy}{2a \sqrt{a}}$.
6. Required the 4th root of $256a^4x^8$.     Anf. $4ax^2$.
7. To find the 5th root of $- 32x^5y^{10}$.     Anf. $- 2xy^2$.

## CASE II.

### *To find the Square Root of a Compound Quantity.*

#### RULE.

THIS is performed like as in numbers, thus :

1. Range the quantities according to the dimenfions of fome letter, and fet the root of the firft term in the quotient.

2. Subtract the fquare of the root thus found, from the firft term, and bring down the two next terms to the re-mainder for a dividend.

3. Divide the dividend by double the root, and fet the refult both in the quotient and divifor.

4. Multiply the divifor, thus increafed, by the term laft put in the quotient, and fubtract the product from the dividend.

And fo on, as in common arithmetic.

#### EXAMPLES.

1. Extract the fquare root of $x^4 - 4x^3 + 6x^2 - 4x + 1$.

$$x^4 - 4x^3 + 6x^2 - 4x + 1 \ ( \ x^2 - 2x + 1 = \text{root}$$
$$x^4$$

$$2x^2 - 2x ) - 4x^3 + 6x^2$$
$$\qquad\qquad - 4x^3 + 4x^2$$

$$2x^2 - 4x + 1 \ ) \ 2x^2 - 4x + 1$$
$$\qquad\qquad\quad 2x^2 - 4x + 1$$

2. Find the root of $4a^4 + 12a^3x + 13a^2x^2 + 6ax^3 + x^4$.

$$4a^4 + 12a^3x + 13a^2x^2 + 6ax^3 + x^4 \ ( \ 2a^2 + 3ax + x^2$$
$$4a^4$$

$$4a^2 + 3ax \ ) \ 12a^3x + 13a^3x^2$$
$$\qquad\qquad\quad 12a^3x + 9a^2x^2$$

$$4a^2 + 6ax + x^2 \ ) \ 4a^2x^2 + 6ax^3 + x^4$$
$$\qquad\qquad\qquad\quad 4a^2x^2 + 6ax^3 + x^4$$

3. Required the fquare root of $a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4$.                    Anf. $a^2 + 2ax + x^2$.

4. Required the fquare root of $x^4 - 2x^3 + \frac{3}{2}x^2 - \frac{x}{2} + \frac{1}{16}$.                    Anf. $x^2 - x + \frac{1}{4}$.

5. It is required to find the fquare root of $a^2 + b^2x^2$.

$$\text{Anf. } a - \frac{b^2x^2}{2a} - \frac{b^4x^4}{8a^3} - \frac{b^6x^6}{16a^5} - \&c.$$

## CASE III.

### To find the Roots of Powers in General.

### RULE *.

1. FIND the root of the firft term, and place it in the quotient.

2. Subtract its power from that term, and bring down the fecond term for a dividend.

3. Involve the root, laft found, to the next lower power, and multiply it by the index of the given power, for a divifor.

4. Divide the dividend by the divifor, and the quotient will be the next term of the root.

5. Involve the whole root to the power to be extracted, then fubtract the power thus arifing from the given power, and always divide the firft term of the remainder by the divifor firft found; and fo on till the whole is finifhed.

### EXAMPLES.

1. To find the fq. root of $a^4 - 2a^3x + 3a^2x^2 - 2ax^3 + a^4$.

$a^4 - 2a^3x + 3a^2x^2 - 2ax^3 + x^4 \ ( a^2 - ax + x^2$

$a^4$

___

$2a^2 ) - 2a^3x$

___

$a^4 - 2a^3x + a^2x^2 = (a^2 - ax)^2$

___

$2a^2 ) 2a^2x^2$

___

$a^4 - 2a^3x + 3a^2x^2 - 2ax^3 + x^4 = (a^2 - ax + x^2)^2.$

___

2. Find

___

* As this method, in high powers, is generally thought too laborious, it may not be improper to obferve, that the roots of compound quantities may fometimes be eafily difcovered, thus :

1. Ex.

2. Find the cube root of $x^6 + 6x^5 - 40x^3 + 96x - 64$.

$x^6 + 6x^5 - 40x^3 + 96x - 64 \ ( \ x^2 + 2x - 4$

$x^6$

———————

$3x^4 \ ) \ 6x^5$

———————

$x^6 + 6x^5 + 12x^4 + 8x^3 = ( x^2 + 2x )^3$

———————

$3x^4 \ ) - 12x^4$

———————

$x^6 + 6x^5 - 40x^3 + 96x - 64 = ( x^2 + 2x - 4 )^3,$

———————

3. Required the square root of $a^2 + 2ab + 2ac + b^2 + 2bc + c^2$.     Anf. $a + b + c$.

4. Required the cube root of $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$.    Anf. $x^2 - 2x + 1$.

5. Required the biquadrate root of $16a^4 - 96a^3x + 216a^2x^2 - 216ax^3 + 81x^4$.    Anf. $2a - 3x$.

6. Required the 5th root of $32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1$.     Anf. $2x - 1$.

7. Required the cube root of $1 - x^3$.

———————————————————

1. Extract the roots of fome of the moft fimple terms, and connect them together by the fign $+$ or $-$, as may be judged moft fuitable for the purpofe.

2. Involve the compound root, thus found, to the proper power; then, if it be the fame with the given quantity, it is the root required.

3. But if it be found to differ only in fome of the figns, change them from $+$ to $-$, or from $-$ to $+$, till its power agrees with the given one throughout.

Thus, in the 5th example, the root $2a - 3x$, is the difference of the roots of the firft and laft terms; and in the 3d example, the root $a + b + c$ is the fum of the roots of the 1ft, 4th, and 6th terms. The fame may alfo be obferved of the 6th example, where the root is found from the firft and laft terms.

       SURDS.

## SURDS.

SURDS are fuch quantities as have no exact root, being ufually expreffed by fractional indices, or by means of the radical fign $\sqrt{}$ .

Thus, $2^{\frac{1}{2}}$, or $\sqrt{2}$, denotes the fquare root of 2 ; and $3^{\frac{2}{3}}$ or $\sqrt[3]{3^2}$, the cube root of the fquare of 3 ; where the numerator fhews the power to which the quantity is to be raifed, and the denominator its root.

### PROBLEM I.

*To Reduce a Rational Quantity to the Form of a Surd.*

#### RULE.

RAISE the quantity to a power equivalent to that denoted by the index of the furd ; then over this new quantity place the radical fign, and it will be of the form required.

#### EXAMPLES.

1. To reduce 3 to the form of the fquare root.
   Firft, $3 \times 3 = 3^2 = 9$ ; then $\sqrt{9}$ is the anfwer.
2. To reduce $2x^2$ to the form of the cube root.
   Firft, $2x^2 \times 2x^2 \times 2x^2 = (2x^2)^3 = 8x^6$ ;

   then $\sqrt[3]{8x^6}$ or $(8x^6)^{\frac{1}{3}}$ is the anfwer.

3. Reduce 5 to the form of the cube root.

   Anf. $(125)^{\frac{1}{3}}$ or $\sqrt[3]{125}$.

4. Reduce $\frac{1}{2}xy$ to the form of the fquare root.

   Anf. $\sqrt{\frac{1}{4} x^2 y^2}$ .

5. Reduce 2 to the form of the 5th root.        Anf. $(32)^{\frac{1}{5}}$

6. Let $a^{\frac{1}{2}}$ be reduced to the form of the 6th root.
7. Reduce $a + b$ to the form of the fquare root,
   and $a - b$ to the form of the cube root.

### PROBLEM II.

*To Reduce Quantities of Different Indices, to other equivalent ones, that fhall have a Common Index.*

#### RULE.

1. DIVIDE the indices of the quantities by the given index, and the quotients will be the new indices for thofe quantities.

2. Over the faid quantities, with their new indices, place the given index, and they will make the equivalent quantities required.

3. A common index may alfo be found by reducing the indices of the quantities to a common denominator, and involving each of them to the power denoted by its numerator.

### EXAMPLES.

1. Reduce $15^{\frac{1}{4}}$ and $9^{\frac{1}{6}}$ to equivalent quantities having the common index $\frac{1}{2}$.

$$\frac{1}{4} \div \frac{1}{2} = \frac{1}{4} \times \frac{2}{1} = \frac{2}{4} = \frac{1}{2} \text{ the 1ft index,}$$
$$\frac{1}{6} \div \frac{1}{2} = \frac{1}{6} \times \frac{2}{1} = \frac{2}{6} = \frac{1}{3} \text{ the 2d index.}$$

Therefore $(15^{\frac{1}{2}})^{\frac{1}{2}}$ and $(9^{\frac{1}{3}})^{\frac{1}{2}}$ are the quantities required.

2. Reduce $a^2$ and $x^{\frac{1}{4}}$ to the fame common index $\frac{1}{3}$.

$$\frac{2}{1} \div \frac{1}{3} = \frac{2}{1} \times \frac{3}{1} = \frac{6}{1} \text{ the 1ft index,}$$
$$\frac{1}{4} \div \frac{1}{3} = \frac{1}{4} \times \frac{3}{1} = \frac{3}{4} \text{ the 2d index.}$$

Therefore $(a^6)^{\frac{1}{3}}$ and $(x^{\frac{3}{4}})^{\frac{1}{3}}$ are the quantities required.

3. Reduce $3^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$ to the common index $\frac{1}{6}$.

Anf. $27^{\frac{1}{6}}$ and $4^{\frac{1}{6}}$.

4. Reduce $a^{\frac{1}{2}}$ and $b^{\frac{1}{4}}$ to the common index $\frac{1}{8}$.

Anf. $(a^4)^{\frac{1}{8}}$ and $(b^2)^{\frac{1}{8}}$.

5. Reduce $a^{\frac{1}{n}}$ and $b^{\frac{1}{m}}$ to the fame radical fign.

Anf. $\sqrt[mn]{a^m}$ and $\sqrt[mn]{b^n}$.

6. Reduce $(a + b)^{\frac{1}{2}}$ and $(a - b)^{\frac{1}{3}}$ to a common index.

7. Reduce $(a + b)^{\frac{1}{3}}$ and $(a - b)^{\frac{1}{4}}$ to a common index.

### PROBLEM III.

*To Reduce Surds to their moft Simple Terms.*

#### RULE*.

FIND the greateft power contained in the given furd, and fet its root before the remaining quantities, with the proper radical fign between them.

---

* When the given furd contains no exact power, it is already in its moft fimple terms.

## EXAMPLES.

1. To reduce $\sqrt{48}$ to its moft fimple terms,

$\sqrt{48} = \sqrt{16 \times 3} = \sqrt{16} \times \sqrt{3} = 4 \times \sqrt{3} = 4\sqrt{3}$, the anfwer.

2. Required to reduce $\sqrt[3]{108}$ to its moft fimple terms.

$\sqrt[3]{108} = \sqrt[3]{27 \times 4} = \sqrt[3]{27} \times \sqrt[3]{4} = 3 \times \sqrt[3]{4} = 3\sqrt[3]{4}$, the anfwer.

3. Reduce $\sqrt{125}$ to its fimpleft terms. 　　　　Anf. $5\sqrt{5}$.

4. Reduce $\sqrt{\frac{50}{147}}$ to its fimpleft terms. 　　　Anf. $\frac{5}{21}\sqrt{6}$.

5. Reduce $\sqrt[3]{243}$ to its fimpleft terms. 　　　Anf. $3\sqrt[3]{9}$.

6. Reduce $\sqrt[3]{\frac{16}{81}}$ to its fimpleft terms. 　　Anf. $\frac{2}{3}\sqrt[3]{18}$.

7. Reduce $\sqrt{98a^2x}$ to its fimpleft terms. 　　Anf. $7a\sqrt{2x}$.

8. Reduce $\sqrt{x^3 - a^2x^2}$ to its moft fimple terms.

9. Reduce $(a^3x + 3a^3x^2)^{\frac{1}{3}}$ to its moft fimple terms.

10. Reduce $(32a^6 - 96a^5x)^{\frac{1}{5}}$ to its moft fimple terms.

## PROBLEM IV.

*To Add Surd Quantities together.*

### RULE.

1. Reduce fuch quantities as have unlike indices to other equivalent ones, having a common index.

2. Bring all fractions to a common denominator, and reduce the quantities to their fimpleft terms, as in the laft problem.

3. Then, if the furd part be the fame in them all, annex it to the fum of the rational parts, with the fign of multiplication, and it will give the total fum required.

But if the furd part be not the fame in all the quantities, they can only be added by the figns $+$ and $-$.

### EXAMPLES.

1. It is required to add $\sqrt{27}$ and $\sqrt{48}$ together.

Firft, $\sqrt{27} = \sqrt{9 \times 3} = 3\sqrt{3}$; and $\sqrt{48} = \sqrt{16 \times 3} = 4\sqrt{3}$; then, $3\sqrt{3} + 4\sqrt{3} = (3 + 4)\sqrt{3} = 7\sqrt{3} =$ fum required.

2. It is required to add $\sqrt[3]{500}$, and $\sqrt[3]{108}$ together.

Firft, $\sqrt[3]{500} = \sqrt[3]{125 \times 4} = 5\sqrt[3]{4}$; and $\sqrt[3]{108} = \sqrt[3]{27 \times 4} = 3\sqrt[3]{4}$;

then, $5\sqrt[3]{4} + 3\sqrt[3]{4} = (5 + 3)\sqrt[3]{4} = 8\sqrt[3]{4} =$ fum required.

3. Re-

3. Required the fum of $\sqrt{72}$ and $\sqrt{128}$.     Anf. $14\sqrt{2}$.

4. Required the. fum of $\sqrt{27}$ and $\sqrt{147}$.     Anf. $10\sqrt{3}$.

5. Required the fum of $\sqrt{\frac{2}{3}}$ and $\sqrt{\frac{27}{50}}$.     Anf. $\frac{19}{30}\sqrt{6}$.

6. Required the fum of $\sqrt[3]{40}$ and $\sqrt[3]{135}$.     Anf. $5\sqrt[3]{5}$.

7. Required the fum of $\sqrt[3]{\frac{1}{4}}$ and $\sqrt[3]{\frac{1}{32}}$.     Anf. $\frac{3}{4}\sqrt[3]{2}$.

8. Required the fum of $2\sqrt{a^2 b}$ and $3\sqrt{64bx^4}$.

9. Required the fum of $9\sqrt{243}$ and $10\sqrt{363}$.

10. Required to find the fum of $a^{\frac{1}{n}}$ and $a^{\frac{1}{m}}$.

11. Required the fum of $\sqrt{27a^4 x}$ and $\sqrt{3a^2 x}$.


## PROBLEM V.

*To Subtract, or find the Difference of, Surd Quantities.*

### RULE,

PREPARE the quantities as in the laft rule ; then the differ-
ence of the rational parts annexed to the common furd, will
give the difference of the furds required.

But if the quantities have no common furd, they can
only be fubtracted by means of the fign —.


### EXAMPLES.

1. Required to find the difference of $\sqrt{448}$ and $\sqrt{112}$.
Firft, $\sqrt{448} = \sqrt{64 \times 7} = 8\sqrt{7}$ ; and $\sqrt{112} = \sqrt{16 \times 7} = 4\sqrt{7}$.
Then $8\sqrt{7} - 4\sqrt{7} = 4\sqrt{7}$ the difference required.

2. Required to find the difference of $192^{\frac{1}{3}}$ and $24^{\frac{1}{3}}$.

Firft, $192^{\frac{1}{3}} = (64 \times 3)^{\frac{1}{3}} = 4 \cdot 3^{\frac{1}{3}}$ ; and $24^{\frac{1}{3}} = (8 \times 3)^{\frac{1}{3}} = 2 \cdot 3^{\frac{1}{3}}$.

Then $4 \cdot 3^{\frac{1}{3}} - 2 \cdot 3^{\frac{1}{3}} = 2 \cdot 3^{\frac{1}{3}}$ the difference required.

3. Required the diff. of $2\sqrt{50}$ and $\sqrt{18}$.     Anf. $7\sqrt{2}$.

4. Required the diff. of $320^{\frac{1}{3}}$ and $40^{\frac{1}{3}}$.     Anf. $2 \cdot 5^{\frac{1}{3}}$.

5. Required the diff. of $\sqrt{\frac{3}{5}}$ and $\sqrt{\frac{5}{27}}$.     Anf. $\frac{4}{45}\sqrt{15}$.

6. Required the diff. of $\sqrt[3]{\frac{2}{3}}$ and $\sqrt[3]{\frac{9}{32}}$.     Anf. $\frac{1}{12}\sqrt[3]{18}$.

7. Find the difference of $\sqrt{80a^4 x}$ and $\sqrt{20a^2 x^3}$.
        Anf. $(4a^2 - 2ax)\sqrt{5x}$.

8. Required the difference of $8\sqrt[3]{a^3 b}$ and $\sqrt[3]{a^6 b}$.

9. Required the difference of $x^{\frac{1}{n}}$ and $x^{\frac{1}{m}}$.

PRO.

## PROBLEM VI.

### *To Multiply Surd Quantities together.*

#### RULE.

REDUCE the furds to the fame index; next multiply the rational quantifies together, and the furds together; then the one product annexed to the other will give the whole product required; which may be reduced to its moft fimple terms by Problem 3.

#### EXAMPLES.

1. Required to find the product of $3\sqrt{8}$ and $2\sqrt{6}$.

Here, $3 \times 2 \times \sqrt{8} \times \sqrt{6} = 6\sqrt{8 \times 6} = 6\sqrt{48} = 6\sqrt{16 \times 3}$
$= 6 \times 4 \times \sqrt{3} = 24\sqrt{3}$, their product required.

2. Required to find the product of $\frac{1}{2}\sqrt[3]{\frac{2}{3}}$ and $\frac{3}{4}\sqrt[3]{\frac{5}{6}}$.

Here $\frac{1}{2} \times \frac{3}{4}\sqrt[3]{\frac{2}{3}} \times \sqrt[3]{\frac{5}{6}} = \frac{3}{8} \times \sqrt[3]{\frac{10}{18}} = \frac{3}{8} \times \sqrt[3]{\frac{15}{27}} = \frac{3}{8} \times \frac{1}{3} \times$
$\sqrt[3]{15} = \frac{3}{24}\sqrt[3]{15} = \frac{1}{8}\sqrt[3]{15}$ the product required.

3. Required the product of $5\sqrt{8}$ and $3\sqrt{5}$.    Anf. $30\sqrt{10}$.

4. Required the product of $\frac{1}{2}\sqrt[3]{6}$ and $\frac{2}{3}\sqrt[3]{18}$.    Anf. $\sqrt[3]{4}$.

5. To find the product of $\frac{2}{3}\sqrt{\frac{1}{8}}$ and $\frac{3}{4}\sqrt{\frac{7}{10}}$.    Anf. $\frac{1}{40}\sqrt{35}$.

6. Required the product of $\sqrt[3]{18}$ and $5\sqrt[3]{4}$.    Anf. $10\sqrt[3]{9}$.

7. Required the product of $a^{\frac{1}{3}}$ and $a^{\frac{2}{3}}$.    Anf. $(a^3)^{\frac{1}{3}}$ or $a$.

8. Required the product of $(x + y)^{\frac{1}{2}}$ and $(x + y)^{\frac{1}{3}}$.

9. Required the product of $x + \sqrt{y}$ and $x - \sqrt{y}$.

10. Required the product of $(a + \sqrt{b})^{\frac{1}{2}}$, and $(a - \sqrt{b})^{\frac{1}{2}}$.

11. Required the product of $x^{\frac{1}{n}}$ and $x^{\frac{1}{m}}$.

12. Required the product of $x^{\frac{1}{m}}$ and $y^{\frac{1}{n}}$.

## PROBLEM VII.

### *To Divide one Surd Quantity by another.*

#### RULE.

REDUCE the furds to the fame index; then take the quotient of the rational quantities, and annex it to the quotient of the furds, and it will give the whole quotient required; which may be reduced to its moft fimple terms as before in multiplication.

## EXAMPLES.

1. It is required to divide $8\sqrt{108}$ by $2\sqrt{6}$.

$8 \div 2 . \sqrt{(108 \div 6)} = 4\sqrt{18} = 4\sqrt{(9 \times 2)} = 4 \times 3\sqrt{2} = 12\sqrt{2}$, the quotient required.

2. It is required to divide $8\sqrt[3]{512}$ by $4\sqrt[3]{2}$.

$8 \div 4 = 2$, and $512^{\frac{1}{3}} \div 2^{\frac{1}{3}} = 256^{\frac{1}{3}} = 4.4^{\frac{1}{3}}$;

therefore $2 \times 4 \times 4^{\frac{1}{3}} = 8.4^{\frac{1}{3}} = 8\sqrt[3]{4}$, is the quot. required.

3. Let $6\sqrt{100}$ be divided by $3\sqrt{2}$.    Anf. $10\sqrt{2}$.

4. Let $4\sqrt[3]{1000}$ be divided by $2\sqrt[3]{4}$.    Anf. $10\sqrt[3]{2}$.

5. Let $\frac{3}{4}\sqrt{\frac{1}{135}}$ be divided by $\frac{2}{3}\sqrt{\frac{1}{5}}$.    Anf. $\frac{3}{8}\sqrt{3}$.

6. Let $\frac{5}{7}\sqrt[3]{\frac{9}{3}}$ be divided by $\frac{2}{5}\sqrt[3]{\frac{3}{4}}$.    Anf. $\frac{25}{21}\sqrt[3]{3}$.

7. Let $\frac{2}{5}\sqrt{a}$, or $\frac{2}{5}a^{\frac{1}{2}}$, be divided by $\frac{3}{4}a^{\frac{1}{3}}$.    Anf. $\frac{8}{15}a^{\frac{1}{6}}$.

8. Divide $a^{\frac{2}{3}}$ by $a^{\frac{4}{3}}$.

9. Let the quantity $x^{\frac{1}{n}}$ be divided by the quantity $x^{\frac{1}{m}}$.

10. Let $x^2 - xd - b + d\sqrt{b}$ be divided by $x - \sqrt{b}$.

## PROBLEM VIII.

*To Involve or Raife Surd Quantities to any Power.*

### RULE.

MULTIPLY the index of the quantity by the index of the power to which it is to be raifed, and to the refult annex the power of the rational parts, and it will give the power required.

### EXAMPLES.

1. It is required to find the fquare of $\frac{2}{3}a^{\frac{1}{3}}$.

Firft, $(\frac{2}{3})^2 = \frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$, and $(a^{\frac{1}{3}})^2 = a^{\frac{1}{3} \times 2} = a^{\frac{2}{3}} = (a^2)^{\frac{1}{3}}$:

theref. $(\frac{2}{3}a^{\frac{1}{3}})^2 = \frac{4}{9}(a^2)^{\frac{1}{3}} = \frac{4}{9}\sqrt[3]{a^2}$, the fquare required.

2. It is required to find the cube of $\frac{5}{7}\sqrt{7}$.

Firft, $(\frac{5}{7})^3 = \frac{5}{7} \times \frac{5}{7} \times \frac{5}{7} = \frac{125}{343}$, and $(7^{\frac{1}{2}})^3 = 7^{\frac{3}{2}} = (7^3)^{\frac{1}{2}}$;

theref. $(\frac{5}{7}\sqrt{7})^3 = \frac{125}{343}(7^3)^{\frac{1}{2}} = \frac{125}{343}(343)^{\frac{1}{2}}$, the cube required.

3. Required the fquare of $3\sqrt[3]{3}$.    Anf. $9\sqrt[3]{9}$.

4. Required the cube of $2^{\frac{1}{2}}$, or $\sqrt{2}$.    Anf. $2\sqrt{2}$.

5. Required the 4th power of $\frac{1}{6}\sqrt{6}$.    Anf. $\frac{1}{36}$.

6. It

6. It is required to find the $n$th power of $a^{\frac{1}{m}}$.

7. It is required to find the fquare of $3 + \sqrt{5}$.

8. It is required to find the cube of $2x - 3\sqrt{y}$.

### PROBLEM IX.

*To Extract the Roots of Surd Quantities.*

#### RULE*.

DIVIDE the index of the given quantity by the index of the root to be extracted; then to the refult annex the root of the rational part, and it will give the root required.

#### EXAMPLES.

1. It is required to find the fquare root of $9\sqrt[3]{3}$.

Firft, $\sqrt{9} = 3$, and $(3^{\frac{1}{3}})^{\frac{1}{2}} = 3^{\frac{1}{3} \div 2} = 3^{\frac{1}{6}}$;

therefore $(9\sqrt[3]{3})^{\frac{1}{2}} = 3.3^{\frac{1}{6}}$, is the fquare root required.

2. It is required to find the cube root of $\frac{1}{8}\sqrt{2}$.

Firft, $\sqrt[3]{\frac{1}{8}} = \frac{1}{2}$, and $(\sqrt{2})^{\frac{1}{3}} = 2^{\frac{1}{2} \div 3} = 2^{\frac{1}{6}}$;

therefore $(\frac{1}{8}\sqrt{2})^{\frac{1}{3}} = \frac{1}{2}.2^{\frac{1}{6}}$, is the cube root required.

3. Required the fquare root of $10^3$.    Anf. $10\sqrt{10}$.

4. Required the cube root of $\frac{8}{27}a^3$.    Anf. $\frac{2}{3}a$.

5. Required the 4th root of $3x^2$.    Anf. $3^{\frac{1}{4}}.x^{\frac{1}{2}}$.

6. It is required to find the $n$th root of $x^{\frac{1}{m}}$.

7. Required the fquare root of $x^2 - 4x\sqrt{a} + 4a$.

---

* The fquare root of a binomial or refidual furd, A + B, or A — B, may be found thus: Take $\sqrt{A^2 - B^2} = D$;

then $\sqrt{A + B} = \sqrt{\dfrac{A + D}{2}} + \sqrt{\dfrac{A - D}{2}}$,

and $\sqrt{A - B} = \sqrt{\dfrac{A + D}{2}} - \sqrt{\dfrac{A - D}{2}}$.

Thus, the fquare root of $8 + 2\sqrt{7} = 1 + \sqrt{7}$; and the fquare root of $3 - \sqrt{8} = \sqrt{2} - 1$: but for the cube, or any higher root, no general rule is given.

# INFINITE SERIES.

An Infinite Series is formed from a vulgar fraction, having a compound denominator, or by extracting the root of a furd quantity ; and is fuch as, being continued, would run on infinitely, in the manner of a decimal fraction.

But, by obtaining a few of the firft terms, the law of the progreffion will be manifeft, fo that the feries may be continued, without actually performing the whole operation.

## PROBLEM I.

*To Reduce Fractional Quantities into Infinite Series.*

### RULE.

DIVIDE the numerator by the denominator, as in common divifion ; and the operation continued, as far as may be thought neceffary, will give the feries required.

### EXAMPLES.

1. To change $\dfrac{ax}{a-x}$ into an infinite feries.

$$a-x \;)\; ax \;..\; \Big(\;\; x + \frac{x^2}{a} + \frac{x^3}{a^2} + \frac{x^4}{a^3} + \text{\&c.}$$

$$\frac{ax - x^2}{x^2}$$

$$\frac{x^2 - \dfrac{x^3}{a}}{\dfrac{x^3}{a}}$$

$$\frac{\dfrac{x^3}{a} - \dfrac{x^4}{a^2}}{\dfrac{x^4}{a^2}}$$

$$\frac{\dfrac{x^4}{a^2} - \dfrac{x^5}{a^3}}{\dfrac{x^5}{a^3}}, \text{\&c.}$$

2. Let

2. Let $\dfrac{1}{1+x}$ be converted into an infinite feries.

$$1 + x \;)\; 1 \;.....\; (\; 1 - x + x^2 - x^3 + x^4 - \&c.$$
$$\underline{1 + x}$$
$$-x$$
$$\underline{-x - x^2}$$
$$x^2$$
$$\underline{x^2 + x^3}$$
$$-x^3$$
$$\underline{-x^3 - x^4}$$
$$\underline{x^4}$$

3. Let $\dfrac{b}{a+x}$ be converted into an infinite feries.

        Anf. $\dfrac{b}{a} \times \left( 1 - \dfrac{x}{a} + \dfrac{x^2}{a^2} - \dfrac{x^3}{a^3} + \&c. \right.$

4. Let $\dfrac{b}{a-x}$ be converted into an infinite feries.

        Anf. $\dfrac{b}{a} \times \left( 1 + \dfrac{x}{a} + \dfrac{x^2}{a^2} + \dfrac{x^3}{a^3} \&c. \right)$

5. Let $\dfrac{1+x}{1-x}$ be converted into an infinite feries.

        Anf. $1 + 2x + 2x^2 + 2x^3 + 2x^4 \&c.$

6. Let $\dfrac{a^2}{(a+x)^2}$ be converted into an infinite feries.

        Anf. $1 - \dfrac{2x}{a} + \dfrac{3x^2}{a^2} - \dfrac{4x^3}{a^3} \&c.$

7. Let $\frac{1}{2}$, or its equal $\dfrac{1}{1+1}$, be converted into an infinite feries.

PROBLEM II.

*To Reduce a Compound Surd into an Infinite Series.*

RULE*.

EXTRACT the root as in common arithmetic, and the operation, continued as far as may be thought neceffary, will give the feries required.

———————————————

* This rule is chiefly of ufe in extracting the fquare root, the operation being too tedious for the higher powers.

                               EXAM-

## EXAMPLES.

1. Extract the root of $a^2 + x^2$ in an infinite series.

$$a^2 + x^2 \left( a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} \right. \&c.$$

$$\underline{a^2}$$

$$2a + \frac{x^2}{2a}) \quad x^2$$

$$\frac{x^2 \quad + \quad \dfrac{x^4}{4a^2}}{}$$

$$2a + \frac{x^2}{a} - \frac{x^4}{8a^3} ) - \frac{x^4}{4a^2}$$

$$\frac{-\dfrac{x^4}{4a^2} - \dfrac{x^6}{8a^4} + \dfrac{x^8}{64a^6}}{}$$

$$2a + \frac{x^2}{a} - \frac{x^4}{4a^2}, \quad \&c.) \quad \frac{x^6}{8a^4} - \frac{x^8}{64a^6}$$

$$\frac{x^6}{8a^4} + \frac{x^8}{16a^6}, \&c.$$

$$\overline{-\frac{5x^8}{64a^6}, \&c.}$$

2. Let $\sqrt{1+1}$ be converted into an infinite series.

Anf. $1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32}$ &c.

3. Let $\sqrt{a^2 - x^2}$ be converted into an infinite series.

Anf. $a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5}$ &c.

4. Let $\sqrt[3]{1 - x^3}$ be converted into an infinite series.

Anf. $1 - \frac{x^3}{3} - \frac{x^6}{9} - \frac{5x^9}{81}$ &c.

5. Let $\sqrt{a^2 + b}$ be converted into an infinite series.

6. Expand $\sqrt{a^2 - 2bx - x^2}$ to an infinite series.

## PROBLEM III.

*To Reduce a Binomial Surd into an Infinite Series; or to Extract any Root of a Binomial.*

### RULE*.

SUBSTITUTE the particular letters of the binomial, with their proper figns, in the following general form, and it will give the root required; obferving that P is the firft term, Q the fecond term divided by the firft, $\frac{m}{n}$ the index of the power or root; and A, B, C, D, &c, the foregoing terms with their proper figns:

$$\overline{P + Q}^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n} AQ + \frac{m-n}{2n} BQ + \frac{m-2n}{3n} CQ + \&c.$$

### EXAMPLES.

1. To extract the fq. root of $r^2 - x^2$, in an infinite feries.

Here $P = r^2$, $Q = -\frac{x^2}{r^2}$, and $\frac{m}{n} = \frac{1}{2}$; therefore

$$P^{\frac{m}{n}} = (r^2)^{\frac{m}{n}} = (r^2)^{\frac{1}{2}} = r = A,$$

$$\frac{m}{n} AQ = \frac{1}{2} \times r \times -\frac{x^2}{r^2} = -\frac{x^2}{2r} = B,$$

$$\frac{m-n}{2n} BQ = \frac{1-2}{4} \times -\frac{x^2}{2r} \times -\frac{x^2}{r^2} = -\frac{x^4}{2.4r^3} = C,$$

$$\frac{m-2n}{3n} CQ = \frac{1-4}{6} \times -\frac{x^4}{2.4r^3} \times -\frac{x^2}{r^2} = -\frac{3x^6}{2.4.6r^5} = D,$$

Hence $r - \frac{x^2}{2r} - \frac{x^4}{2.4r^3} - \frac{3x^6}{2.4.6r^5} - \&c.$ or

$r - \frac{x^2}{2r} - \frac{x^4}{8r^3} - \frac{x^6}{16r^5} - \frac{5x^8}{128r^7}$ &c, is the feries required.

2. To find the value of $\frac{1}{(a+b)^2}$, or its equal $(a+b)^{-2}$, in an infinite feries.

---

* Any furd may be taken from the denominator of a fraction and placed in the numerator, and vice verfa, by only changing the fign of its index. Thus,

$\frac{1}{x^2} = 1 \times x^{-2}$; and $\frac{1}{(a+b)^2} = 1 \times (a+b)^{-2}$ or $(a+b)^{-2}$; and

$\frac{a^2}{(a+x)^2} = a^2 (a+x)^{-2}$; and $\frac{x^{\frac{1}{2}}}{x^{\frac{1}{3}}} = x^{\frac{1}{2}} \times x^{-\frac{1}{3}}$; alfo $\frac{(a^2+x^2)^{\frac{1}{2}}}{(a^2-x^2)^{\frac{1}{2}}}$

$= (a^2 + x^2)^{\frac{1}{2}} \times (a^2 - x^2)^{-\frac{1}{2}}.$

Here

Here $P = a$, $Q = \dfrac{b}{a} = a^{-1}b$, and $\dfrac{m}{n} = \dfrac{-2}{1} = -2$; therefore

$$P^{\frac{m}{n}} = (a)^{-2} = a^{-2} = \dfrac{1}{a^2} = A,$$

$$\dfrac{m}{n}AQ = -2 \times \dfrac{1}{a^2} \times \dfrac{b}{a} = -\dfrac{2b}{a^3} = -2a^{-3}b = B,$$

$$\dfrac{m-n}{2n}BQ = -\dfrac{3}{2} \times -\dfrac{2b}{a^3} \times \dfrac{b}{a} = \dfrac{3b^2}{a^4} = 3a^{-4}b^2 = C,$$

$$\dfrac{m-2n}{3n}CQ = -\dfrac{4}{3} \times \dfrac{3b^2}{a^4} \times \dfrac{b}{a} = -\dfrac{4b^3}{a^5} = -4a^{-5}b^3 = D,$$

Hence $a^{-2} - 2a^{-3}b + 3a^{-4}b^2 - 4a^{-5}b^3 + $ &c, or

$$\dfrac{1}{a^2} - \dfrac{2b}{a^3} + \dfrac{3b^2}{a^4} - \dfrac{4b^3}{a^5} + \dfrac{5b^4}{a^6}$$ &c. is the feries required.

3. To find the value of $\dfrac{r^2}{r+x}$, in an infinite feries.

$$\text{Anf. } r - x + \dfrac{x^2}{r} - \dfrac{x^3}{r^2} + \dfrac{x^4}{r^3}, \text{ \&c.}$$

4. To find the value of $\dfrac{1}{(a^2 - x^2)^{\frac{1}{2}}}$ in an infinite feries.

$$\text{Anf. } \dfrac{1}{a} + \dfrac{x^2}{2a^3} + \dfrac{3x^4}{8a^5} + \dfrac{15x^6}{48a^7}, \text{ \&c.}$$

5. To find the value of $\dfrac{a^2}{(a+x)^2}$ in an infinite feries.

$$\text{Anf. } 1 - \dfrac{2x}{a} + \dfrac{3x^2}{a^2} - \dfrac{4x^3}{a^3} + \dfrac{5x^4}{a^4}, \text{ \&c.}$$

6. To find the value of $(a^2 + b)^{\frac{1}{2}}$ in an infinite feries.

$$\text{Anf. } a + \dfrac{b}{2a} - \dfrac{b^2}{8a^3} + \dfrac{b^3}{16a^5} - \dfrac{5b^4}{128a^7}, \text{ \&c.}$$

7. Find the value of $(a^2 - x^2)^{\frac{1}{5}}$ in an infinite feries.

$$\text{Anf. } a^{\frac{2}{5}} \times (1 - \dfrac{x^2}{5a^2} - \dfrac{2x^4}{25a^4} - \dfrac{6x^6}{125a^6}, \text{ \&c.})$$

8. To find the value of $(a^3 - b)^{\frac{1}{3}}$ in an infinite feries.

$$\text{Anf. } a - \dfrac{b}{3a^2} - \dfrac{b^2}{9a^5} - \dfrac{5b^3}{81a^8} - \dfrac{10b^4}{243a^{11}}, \text{ \&c.}$$

9. To find the fquare root of $\dfrac{a^2 + x^2}{a^2 - x^2}$ in an infinite feries.

$$\text{Anf. } 1 + \dfrac{x^2}{a^2} + \dfrac{x^4}{2a^4} + \dfrac{x^4}{2a^6}, \text{ \&c.}$$

10. Find

**10.** Find the cube root of $\dfrac{a^2}{(a^2 + x^2)^2}$ in an infinite feries.

Anf. $\dfrac{1}{a^{\frac{2}{3}}} \times \left( 1 - \dfrac{2x^2}{3a^2} + \dfrac{5x^4}{9a^4} - \dfrac{40x^6}{81a^6}, \&c. \right)$

**11.** Find the value of $\dfrac{ax}{a^2 - ax + x^2}$ in an infinite feries.

Anf. $\dfrac{x}{a} + \dfrac{x^2}{a^2} - \dfrac{x^4}{a^4} - \dfrac{x^5}{a^5}, \&c.$

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# ARITHMETICAL PROPORTION.

ARITHMETICAL PROPORTION is the relation which two quantities, of the fame kind, bear to each other, in refpect to their difference.

Four quantities are faid to be in Arithmetical Proportion, when the difference between the firft and fecond is equal to the difference between the third and fourth.

Thus, 3, 7, 12, 16, and $a$, $a + b$, $c$, $c + b$, are arithmetically proportional.

Arithmetical Progreffion is when a feries of quantities either increafe or decreafe by the fame common difference.

Thus, 1, 3, 5, 7, 9, 11, &c, and $a$, $a + b$, $a + 2b$, $a + 3b$, $a + 4b$, $a + 5b$, &c, are feries in arithmetical progreffion, whofe common differences are 2 and $b$.

The moft ufeful part of arithmetical proportion is contained in the following theorems:

1. When four quantities are in Arithmetical Proportion, the fum of the two extremes is equal to the fum of the two means.

Thus, if 2, 5, 7, 10, and $a$, $b$, $c$, $d$, be in arithmetical proportion; then is $2 + 10 = 5 + 7$, and $a + d = b + c$.

2. In any continued arithmetical progreffion, the fum of the two extremes is equal to the fum of any two terms which are equally diftant from them.

Thus,

Thus, in the feries 2, 4, 6, 8, 10, 12, &c.

Here $2 + 12 = 4 + 10 = 6 + 8 = 14$.

3. The laft term of any arithmetical feries, is equal to the fum, or difference, of the firft term, and the product of the common difference multiplied by the number of terms lefs one ; according as the feries is increafing or decreafing.

Thus, the 20th term of 2, 4, 6, 8, 10, 12, &c, is $=$ $2 + 2 (20 - 1) = 2 + 2 \times 19 = 2 + 38 = 40$.

And the $n$th term of $a$, $a - x$, $a - 2x$, $a - 3x$, $a - 4x$, &c, is $= a - (n - 1) \times x = a - (n - 1) x$.

4. The fum of any feries of quantities in arithmetical progreffion, is equal to the fum of the two extremes multiplied by half the number of terms.

Thus, the fum of 1, 2, 3, 4, 5, 6, &c, continued to the 20th term, is $= \dfrac{(1 + 20) \times 20}{2} = \dfrac{21 \times 20}{2} = 21 \times 10 = 210$.

And the fum of $n$ terms of $a$, $a+x$, $a+2x$, $a+3x$, to $a+mx$, is $= (a + a + mx) . \dfrac{n}{2} = (a + \tfrac{1}{2} mx) . n = (a + \dfrac{n - 1}{2} x) n$.

### EXAMPLES.

1. The firft term of an increafing arithmetical feries is 3, the common difference 2, and the number of terms 20 ; required the fum of the feries ?

Firft, $3 + 2 \times (20 - 1) = 3 + 2 \times 19 = 3 + 38 = 41$, the laft term.

Then $(3 + 41) \times \dfrac{20}{2} = 44 \times 10 = 440$, the fum required.

2. The firft term of a decreafing arithmetical feries is 100, the common difference 3, and the number of terms 34 ; required the fum of the feries ?

Firft, $100 - 3 . (34 - 1) = 100 - 3 . (33) = 100 - 99 = 1$, the laft term.

And $(100 + 1) \times \dfrac{34}{2} = 101 \times 17 = 1717$, the fum required.

3. Required the fum of the neutral numbers 1, 2, 3, 4, 5, 6, &c, continued to 1000 terms ?    Anf. 500500.

4. *Required the fum of the odd numbers 1, 3, 5, 7, 9, &c, continued to 101 terms ?            Anf. 10201.

5. How many ftrokes do the clocks of Venice, which go on to 24 o'clock, ftrike in the compafs of a day ?    Anf. 300.

6. The firft term of a decreafing arithmetical feries is 10, the common difference $\frac{1}{5}$, and the number of terms 21 ; required the fum of the feries ?            Anf. 140.

7. One hundred ftones being placed on the ground, in a ftraight line, at the diftance of 2 yards from each other ; how far will a perfon travel, who fhall bring them one by one to a bafket, which is placed 2 yards from the firft ftone ?
            Anf. 11 miles and 840 yards.

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# APPLICATION of ARITHMETICAL PROGRESSION to MILITARY AFFAIRS.

## QUESTION I.

A TRIANGULAR † Battalion, confifting of thirty ranks, in which the firft rank is formed of one man only, the
            fecond

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* The fum of any number of terms ($n$) of the arithmetical feries of odd numbers 1, 3, 5, 7, 9, &c, is equal to the fquare ($n^2$) of that number.

That is, if 1, 3, 5, 7, 9, &c, be the numbers, then will 1, $2^2$, $3^2$, $4^2$, $5^2$, &c, be the fums of 1, 2, 3, &c, terms.

For 0 + 1, or the fum of 1 term $= 1^2$, or 1
       1 + 3, or the fum of 2 terms $= 2^2$, or 4
       4 + 5, or the fum of 3 terms $= 3^2$, or 9
       9 + 7, or the fum of 4 terms $= 4^2$, or 16, &c.

Whence it is plain, that, let $n$ be any number whatever, the fum of $n$ terms will be $n^2$.

See more on Arithmetical Proportion in the Arithmetic, p. 113.

† By triangular battalion, is to be underftood, a body of troops ranged in the form of a triangle, in which the ranks exceed each other by an equal number of men : if the firft rank confift of one man only, and the difference between the ranks be alfo one, then
            its

fecond of 3, the third of 5, and fo on: What is the ftrength of fuch a triangular battalion ?

Anfwer, 900 men.

## QUESTION II.

A detachment having 12 fucceffive days to march, with orders to advance the firft day only 2 leagues, the fecond 3½, and fo on, increafing 1½ league each day's march : What is the length of the whole march, and what is the laft day's march ?

Anfwer, the laft day's march is 18½ leagues, and 123 leagues is the length of the whole march.

## QUESTION III.

A brigade * of fappers, having carried on 15 yards of fap the firft night, the fecond only 13 yards, and fo on, decreafing 2 yards every night, till at laft they carried on in one night only 3 yards : What is the number of nights they were employed; and what is the whole length of the fap ?

Anfwer, they were employed 7 nights, and the length of the whole fap was 63 yards.

---

its form is that of an equilateral triangle ; and when the difference between the ranks is more than one, its form may then be an ifofceles or fcalene triangle. The practice of forming troops in this order, which is now laid afide, was formerly held in greater efteem than forming them in a folid fquare, as admitting of a greater front, efpecially when the troops were to make fimply a ftand on all fides.

* A brigade of fappers, confifts generally of 8 men, divided equally into two parties ; and whilft one of thefe parties is advancing the fap, the other is furnifhing the gabions, fafcines, and other neceffary implements : and when the firft party is tired, the fecond takes its place, and fo on, till each man in turn has been at the head of the fap. A fap is a fmall ditch, between 3 and 4 feet in breadth and depth ; and is diftinguifhed from the trench by its breadth only, the trench having between 10 and 15 feet breadth. As an encouragement to fappers, the pay for all the work carried on by the whole brigade, is given to the furvivors.

### QUESTION IV.

A number of gabions * being given to be placed in fix ranks, one above the other, in fuch a manner as that each rank exceeding one another equally, the firft may confift of 4 gabions, and the laft of 9 : What is the number of ga-bions in the fix ranks? and what is the difference between each rank?

Anfwer, the difference between the ranks will be one, and the number of gabions in the fix ranks will be 39.

### QUESTION V.

Two detachments, diftant from each other 37 leagues, and both defigning to occupy an advantageous poft equi-diftant from each others camp, fet out at different times ; the firft detachment increafing every day's march one league and a half, and the fecond detachment increafing each day's march 2 leagues : both the detachments arrive at the fame time ; the firft after five days march, and the fecond after four days march : What is the number of leagues marched by each detachment each day ?

The progreffion $\frac{7}{10}$, $2\frac{2}{10}$, $3\frac{7}{10}$, $5\frac{2}{10}$, $6\frac{7}{10}$, anfwers the con-ditions of the firft detachment : and the progreffion $1\frac{5}{8}$, $3\frac{5}{8}$, $5\frac{5}{8}$, $7\frac{5}{8}$, anfwers the conditions of the fecond detachment.

### QUESTION VI.

A deferter, in his flight, travelling at the rate of 8 leagues a day ; and a detachment of dragoons being fent after him, with orders to march the firft day only 2 leagues, the fecond 5 leagues, the third 8 leagues, and fo on : What is the number of days neceffary for the detachment to overtake the deferter, and what will be the number of leagues marched before he is overtaken ?

Anfwer, 5 days are neceffary to overtake him ; and, con-fequently, 40 leagues will be the extent of the march.

---

* Gabions are bafkets open at both ends, made of ozier twigs, and of a cylindrical form : thofe made ufe of at the trenches are 2 feet wide, and about three feet high ; which being filled with earth, ferve as a fhelter from the enemy's fire : and thofe made ufe of to conftruct batteries, are generally higher and broader. There is another fort of gabion, made ufe of to raife a low parapet : its height is from 1 to 2 feet, and 1 foot wide at top, but fomewhat lefs at bottom, to give room for placing the muzzle of a firelock between them : thefe gabions ferve inftead of fand bags. A fand bag is generally made to contain about a cubical foot of earth.

## QUESTION VII.

A convoy* distant 35 leagues, having orders to join its camp, and to march at the rate of 5 leagues per day; its escort departing at the same time, with orders to march the first day only half a league, and the last day 9½ leagues; and both the escort and convoy arriving at the same time: At what distance is the escort from the convoy at the end of each march?

### OF COMPUTING SHOT OR SHELLS IN A FINISHED PILE.

SHOT and Shells are generally piled in three different forms, called triangular, square, or oblong piles, according as their base is either a triangle, a square, or a rectangle.
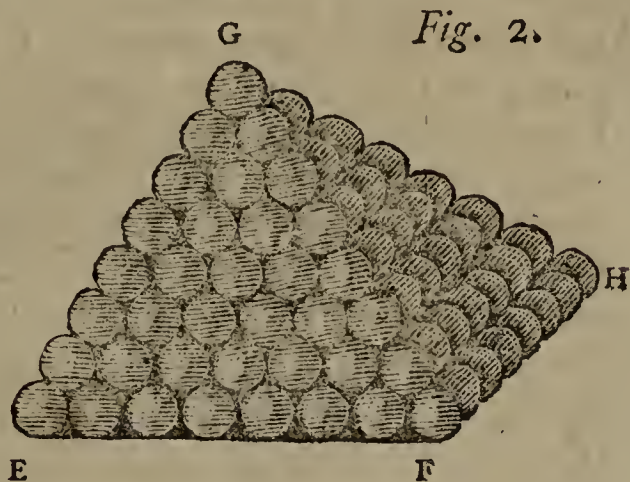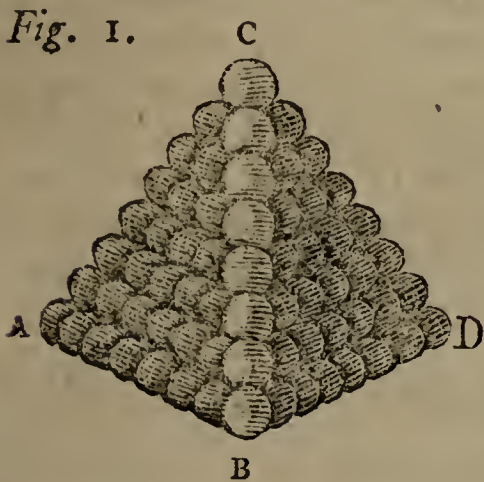


*Fig.* 1.



*Fig.* 2.

ABCD, fig. 1, is a triangular pile,
EFGH, fig. 2, is a square pile.
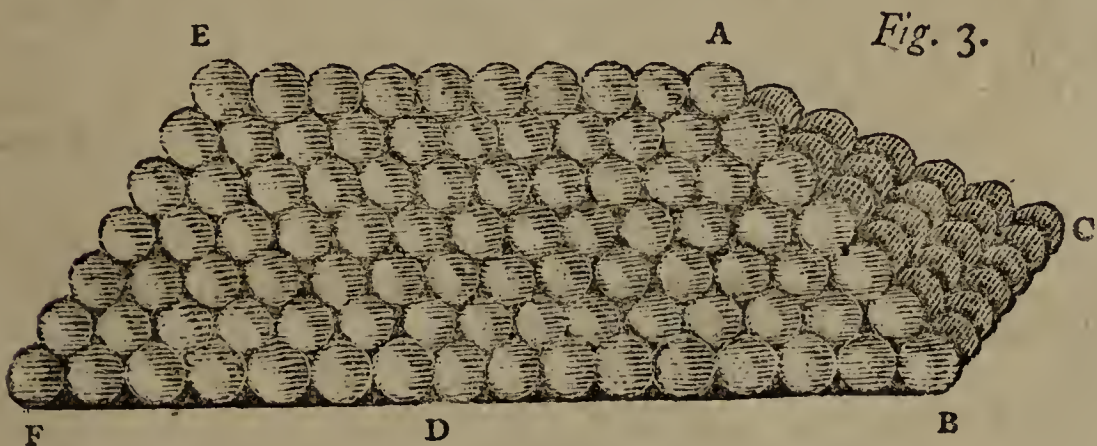


*Fig.* 3.

ABCDEF, fig. 3, is an oblong pile.

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* By convoy is generally meant a supply of ammunition, or provisions, conveyed to a town or army. The body of men that guard this supply, is called escort.

A tri-

A triangular pile is formed by the continual laying of triangular horizontal courfes of fhot one above another, in fuch a manner, as that the fides of thefe courfes, called rows, decreafe by unity from the bottom row to the top row, which ends always in 1 fhot.

A fquare pile is formed by the continual laying of fquare horizontal courfes of fhot one above another, in fuch a manner, as that the fides of thefe courfes decreafe by unity from the bottom to the top row, which ends alfo in 1 fhot.

In the triangular and the fquare piles, the fides or faces being equilateral triangles, the fhot contained in thofe faces form an arithmetical progreffion, having for firft term unity, and for laft term and number of terms, the fhot contained in the bottom row ; for the number of horizontal rows, or the number counted on one of the angles from the bottom to the top, is always equal to thofe counted on one fide in the bottom : the fides or faces in either the triangular or fquare piles, are called arithmetical triangles ; and the numbers contained in thefe, are called triangular numbers ; ABC fig. 1, EFG fig. 2, are arithmetical triangles.

The oblong pile may be conceived as formed from the fquare pile ABCD ; to one fide or face of which, as AD, a number of arithmetical triangles equal to the face have been added : and the number of arithmetical triangles added to the fquare pile, by means of which the oblong pile is formed, is always equal to the fhot in the top row lefs one ; or, which is the fame, the difference between the bottom row of the greater fide and that of the leffer.

### QUESTION VIII.

To find the fhot in the triangular pile ABCD, fig. 1, the bottom row AB confifting of 8 fhot.

### SOLUTION.

The propofed pile confifting of 8 horizontal courfes, each of which forms an equilateral triangle; that is, the fhot contained in thefe being in an arithmetical progreffion, of which the firft and laft term, as alfo the number of terms, are known; it follows, that the fum of thefe particular courfes, or of the 8 progreffions, will be the fhot contained in the propofed pile ; then

The

| | | | | |
|---|---|---|---|---|
| The fhot of the firft or lower triangular courfe will be | $\overline{8+1} \times 4$ | $= 36$ | | |
| the fecond - - - | $\overline{7+1} \times 3\frac{1}{2}$ | $= 28$ | | |
| the third - - - - | $\overline{6+1} \times 3$ | $= 21$ | | |
| the fourth - - - | $\overline{5+1} \times 2\frac{1}{2}$ | $= 15$ | | |
| the fifth - - - | $\overline{4+1} \times 2$ | $= 10$ | | |
| the fixth - - - | $\overline{3+1} \times 1\frac{1}{2}$ | $= 6$ | | |
| the feventh - - - | $\overline{2+1} \times 1$ | $= 3$ | | |
| the eighth - - - | $\overline{1+1} \times \frac{1}{2}$ | $= 1$ | | |

Total - 120 fhot in
the pile propofed.

### QUESTION IX.

To find the fhot of the fquare pile EFGH, fig. 2, the bottom row EF confifting of 8 fhot.

### SOLUTION.

The bottom row containing 8 fhot, and the fecond only 7; that is, the rows forming the progreffion 8, 7, 6, 5, 4, 3, 2, 1, in which each of the terms being the fquare root of the fhot contained in each feparate fquare courfe employed in forming the fquare pile; it follows, that the fum of the fquares of thefe roots will be the fhot required: and the fum of the fquares $\div$ 8. 7. 6. 5. 4. 3. 2. 1, being 204, expreffes the fhot in the propofed pile.

### QUESTION X.

To find the fhot of the oblong pile ABCDEF, fig. 3; in which BF = 16, and BC = 7.

### SOLUTION.

The oblong pile propofed, confifting of the fquare pile ABCD, whofe bottom row is 7 fhot; befides 9 arithmetical triangles or progreffions, in which the firft and laft term, as alfo the number of terms, are known; it follows, that

| | |
|---|---|
| if to the contents of the fquare pile - | 140 |
| we add the fum of the 9th progreffion - | 252 |
| their total gives the contents required - | 392 fhot. |

### REMARK I.

The fhot in the triangular and the fquare piles, as alfo the fhot in each horizontal courfe, may at once be afcer-
tained

tained by the following table: the vertical column A, con-
tains the fhot in the bottom row, from 1 to 20 inclufive;
the column B contains the triangular numbers, or number
of each courfe; the column C contains the fum of the
triangular numbers, that is, the fhot contained in a trian-
gular pile, commonly called pyramidal numbers; the column
D contains the fquare of the numbers of the column A,
that is, the fhot contained in each fquare horizontal courfe;
and the column F contains the fum of thefe fquares or fhot
in a fquare pile.

| C | B | A | D | F |
|---|---|---|---|---|
| Pyramidal numbers. | Triangular numbers. | Natural numbers. | Square of the natural numbers. | Sum of thefe fquare numbers. |
| 1 | 1 | 1 | 1 | 1 |
| 4 | 3 | 2 | 4 | 5 |
| 10 | 6 | 3 | 9 | 14 |
| 20 | 10 | 4 | 16 | 30 |
| 35 | 15 | 5 | 25 | 55 |
| 56 | 21 | 6 | 36 | 91 |
| 84 | 28 | 7 | 49 | 140 |
| 120 | 36 | 8 | 64 | 204 |
| 165 | 45 | 9 | 81 | 285 |
| 220 | 55 | 10 | 100 | 385 |
| 286 | 66 | 11 | 121 | 506 |
| 364 | 78 | 12 | 144 | 650 |
| 455 | 91 | 13 | 169 | 819 |
| 560 | 105 | 14 | 196 | 1015 |
| 680 | 120 | 15 | 225 | 1240 |
| 816 | 136 | 16 | 256 | 1496 |
| 969 | 153 | 17 | 289 | 1785 |
| 1140 | 171 | 18 | 324 | 2109 |
| 1330 | 190 | 19 | 361 | 2470 |
| 1540 | 210 | 20 | 400 | 2870 |

Thus, the bottom row in the triangular pile, confifting of
9 fhot, the contents will be 165; and when of 9 in the fquare
pile, 285.—In the fame manner, the contents either of a
fquare or triangular pile being given, the fhot in the bottom
row may be eafily afcertained.

The contents of any oblong pile by the preceding table
may be alfo with little trouble afcertained, the leffer fide not
exceeding 20 fhot, nor the difference between the leffer and
the greater fide 20: thus, to find the fhot in an oblong pile,
the

the leſſer ſide being 15, and the greater 35, we are firſt to find the contents of the ſquare pile, by means of which the oblong pile may be conceived to be formed; that is, we are to find the contents of a ſquare pile, whoſe bottom row is 15 ſhot; which being 1240, we are, ſecondly, to add theſe 1240 to the product 2400 of the triangular number 120, anſwering to 15, the number expreſſing the bottom row of the arithmetical triangle, multiplied by 20, the number of thoſe triangles; and their ſum, being 3640, expreſſes the number of ſhot in the propoſed oblong pile.

### REMARK II.

The following algebraical expreſſions, deduced from the inveſtigations of the ſums of the powers of numbers in arithmetical progreſſion, which are ſeen upon many gunners callipers*, ſerve to compute with eaſe and expedition the ſhot or ſhells in any pile.

That ſerving to compute any triangular pile is repreſented by

$$\frac{\overline{n+2} \times \overline{n+1} \times n}{6}.$$

That ſerving to compute any ſquare pile, is repreſented by

$$\frac{\overline{n+1} \times \overline{2n+1} \times n}{6}.$$

In each of theſe, the letter $n$ repreſents the number in the bottom row: hence, in a triangular pile, the number in the bottom row being 30; then this pile will be $\overline{30+2} \times \overline{30+1} \times \frac{30}{6} = 4960$ ſhot or ſhells. In a ſquare pile, the number in the bottom row being alſo 30; then this pile will be $\overline{30+1} \times \overline{60+1} \times \frac{30}{6} = 9455$ ſhot or ſhells.

That ſerving to compute any oblong pile, is repreſented by

$$\frac{\overline{2n+1} + 3m \times \overline{n+1} \times n}{6}, \text{ in which the letter } n \text{ denotes}$$

---

* Callipers are large compaſſes, with bowed ſhanks, ſerving to take the diameters of convex and concave bodies. The gunners callipers conſiſt of two thin rules or plates, which are moveable quite round a joint, by the plates folding one over the other: the length of each rule or plate is ſix inches, the breadth about one inch. It is uſual to repreſent, on the plates, a variety of ſcales, tables, proportions, &c, ſuch as are eſteemed uſeful to be known by perſons employed about artillery; but, except the meaſuring of the caliber of ſhot and cannon, and the meaſuring of ſaliant and re-entering angles, none of the articles, with which the callipers are uſually filled, are eſſential to that inſtrument.

the

the number of courfes, and the letter *m* the number of fhot, lefs one, in the top row :-hence, in an oblong pile the number of courfes being 30, and the top row 31 ; this pile will be $\overline{60 + 1} + 90 \times \overline{30 + 1} \times \frac{30}{6} = 23405$ fhot or fhells.

## GEOMETRICAL PROPORTION.

GEOMETRICAL PROPORTION is that relation of two quantities of the fame kind, which arifes from confidering what part the one is of the other, or how often it is contained in it.

When two quantities are compared together, the firft is called the Antecedent, and the fecond the Confequent.

Ratio is the quotient which arifes from dividing the antecedent by the confequent, or the confequent by the antecedent.

Four Quantities are faid to be proportional, when the firft is the fame part or multiple of the fecond, as the third is of the fourth.

Thus, 2, 8, 3, 12, and *a*, *ar*, *b*, *br*, are geometrical proportionals.

Direct Proportion is when the fame relation fubfifts between the firft term and the fecond, as between the third and the fourth.

Thus, 3, 6, 5, 10, and *x*, *ax*, *y*, *ay*, are in direct proportion.

Reciprocal, or Inverfe Proportion, is when one quantity increafes in the fame proportion as another diminifhes.

Thus, 2, 6, 9, 3, and *a*, *ar*, *br*, *b*, are in inverfe proportion.

A Series of Quantities are faid to be in geometrical progreffion, when the firft has the fame ratio to the fecond as the fecond to the third, and the third to the fourth, &c.

Thus, 2, 4, 8, 16, 32, 64, &c, and $a$, $ar$, $ar^2$, $ar^3$, $ar^4$, $ar^5$, &c, are feries in geometrical progreffion.

                                                              The

The moft ufeful part of geometrical proportion, is contained in the following theorems :

1. If four quantities be in geometrical proportion, the product of the two means will be equal to the product of the two extremes.

Thus, if 2, 4, 6, 12, and $a$, $ar$, $b$, $br$, be geometrically proportional, then will $2 \times 12 = 4 \times 6$, and $a \times br = b \times ar$.

2. If four quantities be in geometrical proportion, the rectangle or product of the means divided by either of the extremes, will give the other extreme.

Thus, if 3, 9, 5, 15, and $x$, $ax$, $y$, $ay$, are geometrically proportional, then will $\dfrac{9 \times 5}{3} = 15$, and $\dfrac{ax \times y}{ay} = x$.

And this is the foundation of the Rule-of-Three.

3. In any continued geometrical progreſſion, the product of the two extremes, and that of any other two terms, equally diftant from them, will be equal to each other.

Thus, in the feries 1, 3, 9, 27, 81, 243, &c, it is $1 \times 243 = 3 \times 81 = 9 \times 27 = 243$.

4. In any continued geometrical feries, the laft term, is equal to the firft multiplied by fuch a power of the ratio, as is denoted by the number of terms lefs one.

Thus, in the feries 2, 6, 18, 54, 162, &c; $2 \times 3^4 = 162$.

5. The fum of any feries in geometrical progreſſion, is found by multiplying the laft term by the ratio, and dividing the difference of this product and the firft term by the ratio lefs one.

Thus, the fum of 2, 4, 8, 16, 32, 64, 128, 256, 512, is $\dfrac{512 \times 2 - 2}{2 - 1} = 1024 - 2 = 1022$.

And the fum of $n$ terms of $a$, $ar$, $ar^2$, $ar^3$, $ar^4$, &c, to $ar^{n-1}$, is $\dfrac{ar^{n-1} \times r - a}{r - 1} = \dfrac{ar^n - a}{r - 1} = \dfrac{r^n - 1}{r - 1} a$.

6, If four quantities, $a$, $b$, $c$, $d$, or 2, 6, 5, 15, be proportional ; then will any of the following forms of thofe quantities be alfo proportional, viz.

1. Directly,     $a : b :: c : d$     or  $2 : 6 :: 5 : 15$.
2. Inverfely,   $b : a :: d : c$     or  $6 : 2 :: 15 : 5$.
3. Alternately, $a : c :: b : d$     or  $2 : 6 :: 5 : 15$.

4. Com-

4. Compoundedly, $a : a + b :: c : c + d$ or $2 : 8 :: 5 : 20$.

5. Dividedly, $a : b - a :: c : d - c$ or $2 : 4 :: 5 : 10$.

6. Mixed, $b + a : b - a :: d + c : d - c$ or $8 : 4 :: 20 : 10$.

7. Multiplication, $ra : rb :: c : d$ or $2.3 : 6.3 :: 5 : 15$.

8. Division, $a \div r : b \div r :: c : d$ or $1 : 3 :: 5 : 15$.

9. The numbers $a$, $b$, $c$, $d$, are in harmonical proportion, when $a : d :: a \sim b : c \sim d$; or when their reciprocals $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, $\frac{1}{d}$, are in arithmetical proportion.

### EXAMPLES.

1. The firſt term of a geometrical ſeries is 1, the ratio 2, and the number of terms 10 ; what is the ſum of the ſeries ?

Firſt, $1 \times 2^9 = 1 \times 512 =$ laſt term,

And, $\dfrac{512 \times 2 - 1}{2 - 1} = \dfrac{1024 - 1}{1} = 1023$, the ſum required.

2. The firſt term of a geometric ſeries is $\frac{1}{2}$, the ratio $\frac{1}{3}$, and the number of terms 5 ; required the ſum of the ſeries ?

Firſt, $\frac{1}{2} \times (\frac{1}{3})^4 = \frac{1}{2} \times \frac{1}{81} = \frac{1}{162} =$ the laſt term.

And, $(\frac{1}{2} - \frac{1}{162} \times \frac{1}{3}) \div (1 - \frac{1}{3}) = (\frac{1}{2} - \frac{1}{486}) \div \frac{2}{3} = \frac{121}{243} \times$
$\frac{3}{2} = \frac{121}{81} \times \frac{1}{2} = \frac{121}{162}$, the ſum required.

3. Required the ſum of 1, 3, 9, 27, 81, &c. continued to 12 terms ?                      Anſ. 265720.

4. Required the ſum of 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, $\frac{1}{81}$, &c. continued to 12 terms ?                      Anſ. $\frac{265720}{177147}$.

5. Required the ſum of 1, 2, 4, 8, 16, 32, &c. continued to 100 terms ?

Anſ. 1267650600228229401496703205375.

See more of Geometrical Proportion in the Arithmetic.

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## SIMPLE EQUATIONS.

An Equation, is when two equal quantities, differently expreſſed, are compared together by means of the ſign $=$ placed between them.

Thus, $12 - 5 = 7$ is an equation, expreſſing the equality of the quantities $12 - 5$ and 7.

A Simple

A Simple Equation, is that which contains only one power of the unknown quantity, without including different powers.

Thus, $x - a + b = c$ is a simple equation, containing only one power of the unknown quantity $x$.

Reduction of Equations, is the method of finding the value of the unknown quantity.

It confifts in ordering the equation fo, that the unknown letter or quantity may ftand alone on one fide of the equation, or of the mark of equality, without a co-efficient, and all the reft, or the known quantities, on the other fide.— In general, the unknown quantity is difengaged from the known ones, by performing the reverfe operations. So, if they are connected with it by $+$ or addition, they muft be fubtracted; if by minus ($-$), or fubtraction, they muft be added; if by multiplication, we muft divide by them; if by divifion, we muft multiply; alfo, any power of the unknown quantity is taken away, by extracting the root; and any root is removed, by raifing it to the power. As in the following rules *.

## RULE I.

Any quantity may be tranfpofed from one fide of the equation to the other by changing its fign. And this rule is ufed to remove, or take away quantities, from the fide of the unknown one, when they are connected with it, by the fign $+$ or $-$, or to remove the unknown quantity from them.

Thus, if $x + 3 = 7$; then will $x = 7 - 3 = 4$.
And, if $x - 4 + 6 = 8$; then will $x = 8 + 4 - 6 = 6$.
Alfo, it $x - a + b = c - d$; then will $x = a - b + c - d$.
And, in like manner, if $4x - 8 = 3x + 20$, then will $4x - 3x = 20 + 8$, or $x = 28$.

## RULE II.

When the unknown term is multiplied by any quantity; it is to be taken away by dividing all the terms of the equation by it.

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* Thefe are founded on the general principle of performing equal operations on equal quantities, when it is evident that the refults muft ftill be equal; whether by equal additions, or fubtractions, or multiplications, or divifions, or roots, or powers.

Thus,

Thus, if $ax = ab - a$; then will $x = b - 1$.

And, if $2x + 4 = 16$; then will $x + 2 = 8$, and $x = 8 - 2 = 6$.

In like manner, if $ax + 2ba = 3c^2$;

then will $x + 2b = \dfrac{3c^2}{a}$, and $x = \dfrac{3c^2}{a} - 2b$.

### RULE III.

WHEN the unknown term is divided by any quantity; it may be taken away, by multiplying all the terms of the equation by it.

Thus, if $\dfrac{x}{2} = 5 + 3$; then will $x = 10 + 6 = 16$.

And, if $\dfrac{x}{a} = b + c - d$; then will $x = ab + ac - ad$.

In like manner, if $\dfrac{2x}{3} - 2 = 6 + 4$; then will $2x - 6 =$

$18 + 12$, and $2x = 18 + 12 + 6 = 36$, or $x = \dfrac{36}{2} = 18$.

### RULE IV.

THE unknown quantity in any equation may be made free from surds, by transposing the rest of the terms by Rule 1, and then involving each side to such a power as is denoted by the index of the surd.

Thus, if $\sqrt{x} - 2 = 6$; then will $\sqrt{x} = 6 + 2 = 8$, and $x = 8^2 = 64$.

And, if $\sqrt{4x + 16} = 12$, then will $4x + 16 = 144$, or $4x = 144 - 16 = 128$; and if both sides of the equation be divided by 4, $x$ will be $= 32$.

In like manner, if $\sqrt[3]{2x + 3} + 4 = 8$; then will $\sqrt[3]{2x + 3} = 8 - 4 = 4$, and $2x + 3 = 4^3 = 64$, and $2x = 64 - 3 = 61$, or $x = \dfrac{61}{2} = 30\frac{1}{2}$.

### RULE V.

IF that side of the equation which contains the unknown quantity be a complete power; it may be reduced, by extracting the root of the said power on both sides of the equation.

Thus, if $x^2 + 6x + 9 = 25$; then will $x + 3 = \sqrt{25} = 5$, or $x = 5 - 3 = 2$.

And,

And, if $3x^2 - 9 = 21 + 3$; then will $3x^2 = 21 + 3 + 9$ $= 33$, and $x^2 = \dfrac{33}{3} = 11$, or $x = \sqrt{11}$.

In like manner, if $\dfrac{2x^2}{3} + 10 = 20$; then will $2x^2 + 30$ $= 60$, and $x^2 + 15 = 30$, or $x^2 = 30 - 15 = 15$, or $x = \sqrt{15}$.

## RULE VI.

ANY analogy or proportion may be converted into an equation, by making the product of the two mean terms equal to that of the two extremes.

Thus, if $3x : 16 :: 5 : 10$; then will $3x \times 10 = 16 \times 5$, or $30x = 80$, or $x = \dfrac{80}{30} = \dfrac{8}{3} = 2\frac{2}{3}$

And, if $\dfrac{2x}{3} : a :: b : c$; then will $\dfrac{2cx}{3} = ab$, and $2cx = 3ab$, or $x = \dfrac{3ab}{2c}$.

In like manner, if $12 - x : \dfrac{x}{2} :: 4 : 1$; then will $12 - x = \dfrac{4x}{2} = 2x$, and $2x + x = 12$, or $3x = 12$, and $x = \dfrac{12}{3} = 4$.

## RULE VII.

IF the same quantity be found on both sides of an equation, with the same sign; it may be taken away from each; and if every term in an equation be multiplied or divided by the same quantity, it may be struck out of them all.

Thus, if $4x + a = b + a$; then will $4x = b$, and $x = \dfrac{b}{4}$.

And, if $3ax + 5ab = 8ac$; then will $3x + 5b = 8c$, and $x = \dfrac{8c - 5b}{3}$.

In like manner, if $\dfrac{2x}{3} - \dfrac{8}{3} = \dfrac{16}{3} - \dfrac{8}{3}$; then will $2x = 16$, and $x = 8$.

## MISCELLANEOUS EXAMPLES.

1. Given $5x - 15 = 2x + 6$; to find the value of $x$.

First, $5x - 2x = 6 + 15$

Or, $3x = 6 + 15 = 21$

Therefore $x = \dfrac{21}{3} = 7$

2 Given

2. Given $40 - 6x - 16 = 120 - 14x$; to find $x$.

First, $14x - 6x = 120 - 40 + 16.$

Or $8x = 136 - 40 = 96.$

And therefore $x = \dfrac{96}{8} = 12$

3. Let $5ax - 3b = 2dx + c$ be given; to find $x$.

First, $5ax - 2dx = c + 3b$

Or $(5a - 2d) \times x = c + 3b$

And therefore $x = \dfrac{c + 3b}{5a - 2d}$

4. Let $3x^2 - 10x = 8x + x^2$ be given; to find $x$.

First, $3x - 10 = 8 + x$, by dividing by $x$

Or $3x - x = 8 + 10 = 18$

And therefore $2x = 18$, or $x = \dfrac{18}{2} = 9$

5. Given $6ax^3 - 12abx^2 = 3ax^3 + 6ax^2$; to find $x$.

First, dividing the whole by $3ax^2$

we shall have $2x - 4b = x + 2$

Or $2x - x = 2 + 4b$

That is, $x = 2 + 4b$

6. Let $\dfrac{x}{2} - \dfrac{x}{3} + \dfrac{x}{4} = 10$ be given, to find $x$.

First, $x - \dfrac{2x}{3} + \dfrac{2x}{4} = 20$

Also, $3x - 2x + \dfrac{6x}{4} = 60$

And $12x - 8x + 6x = 240$

Therefore $10x = 240$

And $x = \dfrac{240}{10} = 24$

7. Given $\dfrac{x - 3}{2} + \dfrac{x}{3} = 20 - \dfrac{x + 19}{2}$; to find $x$.

First, $x - 3 + \dfrac{2x}{3} = 40 - x + 19$

Also, $3x - 9 + 2x = 120 - 3x + 57$

Therefore, $3x + 2x + 3x = 120 + 57 + 9$

That is, $8x = 186$, or $x = \dfrac{186}{8} = 23\frac{1}{4}$

8. Let

8. Let $\sqrt{\frac{2x}{3}} + 5 = 7$, be given; to find $x$.

Firſt, $\sqrt{\frac{2x}{3}} = 7 - 5 = 2$

Whence $\frac{2x}{3} = 2^2 = 4$

And $2x = 12$, or $x = \frac{12}{2} = 6$.

9. Let $x + \sqrt{a^2 + x^2} = \frac{2a^2}{\sqrt{a^2 + x^2}}$, be given; to find $x$.

Firſt, $x\sqrt{a^2 + x^2} + a^2 + x^2 = 2a^2$

Whence $x\sqrt{a^2 + x^2} = a^2 - x^2$

And $x^2 \times \overline{a^2 + x^2} = \overline{a^2 - x^2}^2 = a^4 - 2a^2x^2 + x^4$

Or $a^2x^2 + x^4 = a^4 - 2a^2x^2 + x^4$

Or $a^2x^2 + 2a^2x^2 = a^4$, or $3a^2x^2 = a^4$

Conſeq. $x^2 = \frac{a^4}{3a^2} = \frac{a^2}{3}$ and $x = \sqrt{\frac{a^2}{3}} = a\sqrt{\frac{1}{3}}$.

### EXAMPLES FOR PRACTICE.

1. Given $3y - 2 + 24 = 31$; to find $y$.  Anſ. $y = 3$.

2. Given $x + 18 = 3x - 5$; to find $x$.  Anſ. $x = 11\frac{1}{2}$.

3. Given $6 - 2x + 10 = 20 - 3x - 2$; to find $x$. Anſ. $x = 2$.

4. Given $x + \frac{1}{2}x + \frac{1}{3}x = 11$; to find $x$.  Anſ. $x = 6$.

5. Given $2x - \frac{1}{2}x + 1 = 5x - 2$; to find $x$. Anſ. $x = \frac{6}{7}$.

6. Given $3ax + \frac{a}{2} - 3 = bx - a$; to find $x$.

Anſ. $x = \frac{6 - 3a}{6a - 2b}$.

7. Given $\frac{1}{2}x + \frac{1}{3}x - \frac{1}{4}x = \frac{1}{2}$; to find $x$.  Anſ. $x = \frac{6}{7}$.

8. Given $\sqrt{12 + x} = 2 + \sqrt{x}$; to find $x$.  Anſ. $x = 4$.

9. Given $x + a = \frac{x^2}{a + x}$; to determine $x$. Anſ. $x = -\frac{a}{2}$.

10. Given $\sqrt{a^2 + x^2} = \sqrt[4]{b^4 + x^4}$ : to find $x$.

Anſ. $x = \sqrt{\frac{b^4 - a^4}{2a^2}}$.

11. Given $\sqrt{x} + \sqrt{a + x} = \frac{2a}{\sqrt{a + x}}$; to find $x$.

Anſ. $x = \frac{a}{3}$.

VOL. I.  R  12. Given

12. Given $\dfrac{a}{1+x} + \dfrac{a}{1-x} = b$; to find $x$.

$$\text{Anf. } x = \sqrt{\dfrac{b-2a}{b}}.$$

13. Given $a + x = \sqrt{a^2 + x}\sqrt{b^2 + x^2}$; to find $x$.

$$\text{Anf. } x = \dfrac{b^2}{4a} - a.$$

## OF REDUCING DOUBLE, TRIPLE, &c, EQUATIONS, CONTAINING TWO, THREE, OR MORE UNKNOWN QUANTITIES.

### PROBLEM I.

*To Exterminate Two Unknown Quantities; Or, to Reduce the Two Simple Equations containing them, to a Single one.*

### RULE I.

1. Observe which of the unknown quantities is the leaft involved, and find its value in each of the equations, by the methods already explained.

2. Let the two values, thus found, be made equal to each other, and there will arife a new equation with only one unknown quantity in it, whofe value may be found as before.

### EXAMPLES.

1. Given $\left\{ \begin{array}{l} 2x + 3y = 23 \\ 5x - 2y = 10 \end{array} \right\}$; to find $x$ and $y$.

From the firft equation $x = \dfrac{23 - 3y}{2}$

And from the fecond $x = \dfrac{10 + 2y}{5}$

Confequently $\dfrac{23 - 3y}{2} = \dfrac{10 + 2y}{5}$

Or $115 - 15y = 20 + 4y$

Or $19y = 115 - 20 = 95$

That is $y = \dfrac{95}{19} = 5$

And $x = \dfrac{23 - 15}{2} = 4$

2. Given

2. Given $\left\{\begin{array}{l} x + y = a \\ x - y = b \end{array}\right\}$ ; to find $x$ and $y$.

From the firſt equation $x = a - y$

And from the ſecond $x = b + y$

Therefore $a - y = b + y$, or $2y = a - b$

Conſeq. $y = \dfrac{a - b}{2}$, and $x = a - y$

Or $x = a - \dfrac{a - b}{2} = \dfrac{a + b}{2}$.

3. Given $\left\{\begin{array}{l} \frac{1}{2}x + \frac{1}{3}y = 7 \\ \frac{1}{3}x + \frac{1}{2}y = 8 \end{array}\right\}$ ; to find $x$ and $y$

From the firſt equation $x = 14 - \dfrac{2y}{3}$

And from the ſecond $x = 24 - \dfrac{3y}{2}$

Therefore $14 - \dfrac{2y}{3} = 24 - \dfrac{3y}{2}$

And $\qquad 42 - 2y = 72 - \dfrac{9y}{2}$

Or $\qquad 84 - 4y = 144 - 9y$

Whence $\quad 5y = 144 - 84 = 60$

Therefore $y = \dfrac{60}{5} = 12$

And $\qquad x = 14 - \dfrac{2y}{3} = 14 - \dfrac{24}{3} = 6$.

4. Given $4x + y = 34$, and $4y + x = 16$ ; to find $x$ and $y$.
$\qquad$ Anf. $x = 8$, and $y = 2$.

5. Given $\dfrac{2x}{5} + \dfrac{3y}{4} = \dfrac{9}{20}$, and $\dfrac{3x}{4} + \dfrac{2y}{5} = \dfrac{61}{120}$ ; to find $x$
and $y$. $\qquad$ Anf. $x = \frac{1}{2}$, and $y = \frac{1}{3}$.

6. Given $x + y = s$, and $x^2 - y^2 = d$ ; to find $x$ and $y$.
$\qquad$ Anf. $x = \dfrac{s^2 + d}{2s}$, and $y = \dfrac{s^2 - d}{2s}$.

7. Given $x - y = d$, and $x : y :: n : m$ ; to find $x$ and $y$.

### RULE II.

1. CONSIDER which of the unknown quantities you would firſt exterminate, and let its value be found in that equation where it is leaſt involved.

2. Subſtitute the value, thus found, for its equal in the other equation, and there will ariſe a new equation, with

R 2 $\qquad\qquad\qquad\qquad\qquad$ only

only one unknown quantity, whofe value may be found as before.

<div align="center">E X A M P L E S.</div>

1. Given $\left\{ \begin{array}{l} x + 2y = 17 \\ 3x - y = 2 \end{array} \right\}$ ; to find $x$ and $y$.

From the firft equation $x = 17 - 2y$.
This value fubftituted for $x$ in the fecond,
Gives $(17 - 2y) \times 3 - y = 2$
Or      $51 - 6y - y = 2$, or $51 - 7y = 2$
That is $7y = 51 - 2 = 49$

Hence $y = \dfrac{49}{7} = 7$, and $x = 17 - 2y = 17 - 14 = 3$.

2. Given $\left\{ \begin{array}{l} x + y = 13 \\ x - y = 3 \end{array} \right\}$ ; to find $x$ and $y$.

From the firft equation $x = 13 - y$
This value being fubftituted for $x$ in the 2d,
Gives $13 - y - y = 3$, or $13 - 2y = 3$
That is $2y = 13 - 3 = 10$

Hence $y = \dfrac{10}{2} = 5$, and $x = 13 - y = 13 - 5 = 8$.

3. Given $\left\{ \begin{array}{l} a : b :: x : y \\ x^2 + y^2 = c \end{array} \right\}$ ; to find $x$ and $y$.

The firft analogy turned into an equation,

gives $bx = ay$, or $x = \dfrac{ay}{b}$

And this value of $x$ being fubftituted in the 2d, gives

$(\dfrac{ay}{b})^2 + y^2 = c$, or $\dfrac{a^2 y^2}{b^2} + y^2 = c$

Theref. $a^2 y^2 + b^2 y^2 = b^2 c$, or $y^2 = \dfrac{b^2 c}{a^2 + b^2}$

Hence $y = \sqrt{\dfrac{b^2 c}{a^2 + b^2}}$, and $x = \sqrt{\dfrac{a^2 c}{a^2 + b^2}}$.

4. Given $2x + 3y = 16$, and $3x - 2y = 11$ ; to find $x$ and $y$.                      Anf. $x = 5$, and $y = 2$.

5. Given $\dfrac{x}{7} + 7y = 99$, and $\dfrac{y}{7} + 7x = 51$ ; to find $x$ and $y$.
                      Anf. $x = 7$, and $y = 14$.

6. Given $\dfrac{x}{2} - 12 = \dfrac{y}{4} + 8$, and $\dfrac{x+y}{5} + \dfrac{x}{3} - 8 = \dfrac{2y - x}{4}$
$+ 27$ ; to find $x$ and $y$.          Anf. $x = 60$, and $y = 40$.

7. Given $a : b :: x : y$, and $x^3 - y^3 = a$; to find $x$ and $y$.

$$\text{Anf. } x = \sqrt[3]{\frac{ba^3}{a^3 - b^3}}, \text{ and } y = \sqrt[3]{\frac{ab^3}{a^3 - b^3}}.$$

### RULE III.

1. Let the given equations be multiplied, or divided, by fuch numbers or quantities as will make the term which contains one of the unknown quantities the fame in both equations.

2. Then, by adding or fubtracting the equations, according as the cafe may require, there will arife a new equation, with only one unknown quantity, as before.

### EXAMPLES.

1. Given $\left\{ \begin{matrix} 3x + 5y = 40 \\ x + 2y = 14 \end{matrix} \right\}$; to find $x$ and $y$.

Firft, multiply the 2d equation by 3,
and it will give $3x + 6y = 42$.
Then fubtract the firft from the laft equation,
and it will give $6y - 5y = 42 - 40$, or $y = 2$,
and therefore $x = 14 - 2y = 14 - 4 = 10$.

2. Given $\left\{ \begin{matrix} 5x - 3y = 9 \\ 2x + 5y = 16 \end{matrix} \right\}$; to find $x$ and $y$.

Let the firft equation be multiplied by 2, and the 2d by 5, and we fhall have $10x - 6y = 18$
and $10x + 25y = 80$

Then if the former of thefe be fubtracted from the latter

it will give $31y = 62$, or $y = \dfrac{62}{31} = 2$

Confeq. $x = \dfrac{9 + 3y}{5}$, by the firft equation

Or $x = \dfrac{9 + 6}{5} = \dfrac{15}{5} = 3$.

### Another Method.

Multiply the firft equation by 5, and the fecond by 3,
and we fhall have $\left\{ \begin{matrix} 25x - 15y = 45 \\ 6x + 15y = 48 \end{matrix} \right.$

Now, let thefe two equations be added together,

and the fum will be $31x = 93$, or $x = \dfrac{93}{31} = 3$.

Confeq.

Confeq. $y = \dfrac{16 - 2x}{5}$, by the fecond equation,

Or $y = \dfrac{16 - 6}{5} = \dfrac{10}{5} = 2$, as before.

### MISCELLANEOUS EXAMPLES.

1. Given $\dfrac{x + 2}{3} + 8y = 31$, and $\dfrac{y + 5}{4} + 10x = 192$; to find $x$ and $y$.          Anf. $x = 19$, and $y = 3$.

2. Given $\dfrac{2x - y}{2} + 14 = 18$, and $\dfrac{2y + x}{3} + 16 = 19$; to find $x$ and $y$.          Anf. $x = 5$, and $y = 2$.

3. Given $\dfrac{2x + 3y}{6} + \dfrac{x}{3} = 8$, and $\dfrac{7y - 3x}{2} - y = 11$; to find $x$ and $y$.          Anf. $x = 6$, and $y = 8$.

4. Given $ax + by = c$, and $dx + ey = f$; to find $x$ and $y$.
$$\text{Anf. } x = \frac{ce - bf}{ae - bd}, \text{ and } y = \frac{af - dc}{ae - bd}.$$

### PROBLEM II.

*To Exterminate Three Unknown Quantities ; Or, to Reduce the Three Simple Equations, containing them, to a Single one.*

### RULE.

1. Let $x$, $y$, and $z$, be the three unknown quantities, to be exterminated.

2. Find the value of $x$ from each of the three given equations.

3. Compare the firft value of $x$ with the fecond, and an equation will arife involving only $y$ and $z$.

4. In like manner, compare the firft value of $x$ with the third, and another equation will arife involving only $y$ and $z$.

5. Find the values of $y$ and $z$ from thefe two equations, according to the former rules ; and $x$, $y$, and $z$, will be exterminated as required.

*Note.* Much in the fame manner may any number of unknown quantities be exterminated. But there are often fhorter methods for performing the operation, which will be beft learnt from practice.

### EXAMPLES.

1. Given $\begin{cases} x + y + z = 29 \\ x + 2y + 3z = 62 \\ \frac{1}{2}x + \frac{1}{3}y + \frac{1}{4}z = 10 \end{cases}$ ; to find $x$, $y$, and $z$.

From the firſt,   $x = 29 - y - z$

From the ſecond, $x = 62 - 2y - 3z$

From the third,  $x = 20 - \frac{2}{3}y - \frac{1}{2}z$

Whence $29 - y - z = 62 - 2y - 3z$

And    $29 - y - z = 20 - \frac{2}{3}y - \frac{1}{2}z$

Alſo from the firſt of theſe, $y = 33 - 2z$

And from the ſecond,    $y = 27 - \frac{3}{2}z$

Theref. $33 - 2z = 27 - \frac{3}{2}z$, or $z = 12$

Whence alſo $y = 33 - 2z = 9$

And $x = 29 - y - z = 8$.

2. Given $\begin{cases} \frac{1}{2}x + \frac{1}{3}y + \frac{1}{4}z = 62 \\ \frac{1}{3}x + \frac{1}{4}y + \frac{1}{5}z = 47 \\ \frac{1}{4}x + \frac{1}{5}y + \frac{1}{6}z = 38 \end{cases}$ ; to find $x$, $y$, and $z$.

Firſt, the given equations, cleared of fractions, become

$$12x + 8y + 6z = 1488$$
$$20x + 15y + 12z = 2820$$
$$30x + 24y + 20z = 4560$$

Then, if the ſecond of theſe equations be ſubtracted from double the firſt, and 3 times the third from 5 times the ſecond, we ſhall have

$$4x + y = 156$$
$$10x + 3y = 420$$

And again, if the ſecond of theſe be ſubtracted from 3 times the firſt, it will give

$$12x - 10x = 468 - 420, \text{ or } x = \frac{48}{2} = 24,$$

Theref. $y = 156 - 4x = 60$, and $z = \dfrac{1488 - 8y - 12x}{6} = 120$.

3. Given $x + y + z = 53$, and $x + 2y + 3z = 105$, and $x + 3y + 4z = 134$; to find $x$, $y$, and $z$.
          Anſ. $x = 24$, $y = 6$, and $z = 23$.

4. Given $x + y = a$, $x + z = b$, and $y + z = c$; to find $x$, $y$, and $z$.

5. Given $\begin{cases} ax + by + cz = m \\ dx + ey + fz = n \\ gx + hy + kz = p \end{cases}$ ; to find $x$, $y$, and $z$.

A COL-

## A COLLECTION OF QUESTIONS PRODUCING SIMPLE EQUATIONS.

1. To find two numbers, such, that their sum shall be 40, and their difference 16.

Let $x$ denote the least of the two numbers required,

Then will $x + 16 =$ to the greater,
And $x + x + 16 = 40$ by the question,
That is, $2x = 40 - 16 = 24$

Or $x = \dfrac{24}{2} = 12$, the least number,

And $x + 16 = 12 + 16 = 28$, the greater number required.

2. What number is that, whose $\frac{1}{3}$ part exceeds its $\frac{1}{4}$ part by 16 ?

Let $x =$ number required,
Then will its $\frac{1}{3}$ part be $\frac{1}{3}x$, and its $\frac{1}{4}$ part $\frac{1}{4}x$;
Therefore $\frac{1}{3}x - \frac{1}{4}x = 16$ by the question,
That is, $x - \frac{3}{4}x = 48$, or $4x - 3x = 192$;
Hence $x = 192$, the number required.

3. Divide 1000l. among A, B, and C, so that A shall have 72l. more than B, and C 100l. more than A.

Let $x =$ B's share of the given sum,
Then will $x + 72 =$ A's share,
And $x + 172 =$ C's share,
The sum of all their shares is $x + x + 72 + x + 172$,
Or $3x + 244 = 1000$ by the question,
That is, $3x = 1000 - 244 = 756$,

Or $x = \dfrac{756}{3} = 252l. =$ B's share;

Hence $x + 72 = 252 + 72 = 324l. =$ A's share,
And $x + 172 = 252 + 172 = 424l. =$ C's share,
      B's share,     252l.
      A's share,     324l.
      C's share,     424l.
Sum of all,      1000l. the proof.

4. A prize of 1000l. is to be divided between two persons, whose shares of it are in the proportion of 7 to 9; required the share of each ?

Let $x =$ first person's share,
Then will $1000 - x =$ second person's share,
And $x : 1000 - x :: 7 : 9$, by the question,

That

That is, $9x = (1000 - x) \times 7 = 7000 - 7x$,
Or $\quad 9x + 7x = 16x = 7000$,

Hence $x = \dfrac{7000}{16} = 437l.\ 10s. = $ 1ft fhare,

And $1000 - x = 1000 - 437l.\ 10s. = 562l.\ 10s.$ 2d fhare.

5. The paving of a fquare at 2s. a yard, coft as much as the inclofing it at 5s. a yard: required the fide of the fquare?

    Let $x = $ fide of the fquare fought,
    Then $4x = $ yards of inclofure,
    And $x^2 = $ yards of pavement;
    Hence $4x \times 5 = 20x = $ price of inclofing,
    And $x^2 \times 2 = 2x^2 = $ price of paving,
    But $2x^2 = 20x$ by the queftion,
Theref. $2x = 20$, and $x = 10 = $ length of the fide required.

6. A labourer engaged to ferve for 40 days, on thefe conditions; that for every day he worked, he was to receive 20d. but for every day he played, or was abfent, he was to forfeit 8d. Now at the end of the time he had to receive 1l. 11s. 8d. It is required to find how many days he worked, and how many he was idle?

    Let $x$ be the number of days he worked,
    Then will $40 - x$ be the number of days he was idle,
    Alfo $x \times 20 = 20x = $ the fum earned,
    And $(40 - x) \times 8 = 320 - 8x = $ fum forfeited,
    Hence $20x - (320 - 8x) = 380d. = $ (1l. 11s. 8d.) by
      the queftion; that is, $20x - 320 + 8x = 380$,
    Or $28x = 380 + 320 = 700$,

    Hence $x = \dfrac{700}{28} = 25 = $ number of days he worked,

    And $40 - x = 40 - 25 = 15 = $ number of days he
    was idle.

7. Out of a cafk of wine, which had leaked away $\frac{1}{3}$, 21 gallons were drawn; and then, being gauged, it appeared to be half full; how much did it hold?

    Let it be fuppofed to have held $x$ gallons,
    Then it would have leaked $\frac{1}{3}x$ gallons,
    Confeq. there had been taken away $21 + \frac{1}{3}x$ gallons.
    But $21 + \frac{1}{3}x = \frac{1}{2}x$ by the queftion,
    That is, $63 + x = \frac{3}{2}x$
    Or $\quad 126 + 2x = 3x$
    Hence $3x - 2x = 126$
    Or $x = 126 = $ number of gallons required.

                      8, What

8. What fraction is that, to the numerator of which, if 1 be added, the value will be $\frac{1}{3}$; but if 1 be added to the denominator, its value will be $\frac{1}{4}$?

Let the fraction be reprefented by $\frac{x}{y}$.

Then will $\dfrac{x+1}{y} = \dfrac{1}{3}$,

And $\qquad \dfrac{x}{y+1} = \dfrac{1}{4}$,

Hence $\qquad 3x + 3 = y$,

And $\qquad 4x = y + 1$,

Theref. $4x - 3x - 3 = y + 1 - y$,

That is $x - 3 = 1$,

Or $x = 4$; and hence $y = 3x + 3 = 12 + 3 = 15$.

So that $\frac{4}{15}$ is the fraction required.

9. A market woman bought in a certain number of eggs at 2 a penny, and as many more at 3 a penny, and fold them all out again at the rate of 5 for two-pence, and by fo doing loft 4d. what number of eggs had fhe?

Let $x =$ number of eggs of each fort,

Then will $\frac{1}{2}x =$ price of the firft fort,

And $\frac{1}{3}x =$ price of the fecond fort;

But $5 : 2 :: 2x$ (the whole number of eggs) $: \frac{4}{5}x$;

Whence $\frac{4}{5}x =$ price of both forts, at 5 for 2 pence,

And $\frac{1}{2}x + \frac{1}{3}x - \frac{4}{5}x = 4$ by the queftion;

That is $x + \frac{2}{3}x - \frac{8}{5}x = 8$.

Or $\qquad 3x + 2x - \frac{24}{5}x = 24$;

Or $\qquad 15x + 10x - 24x = 120$,

Or $x = 120 =$ number of eggs of each fort.

10. If A can do a piece of work alone in 10 days, and B in 13; fet them both about it together, in what time will it be finifhed?

Let the time fought be denoted by $x$.

Then 10 days : 1 work :: $x$ days : $\frac{1}{10}x$.

And 13 days : 1 work :: $x$ days : $\frac{1}{13}x$:

Hence $\frac{1}{10}x =$ part done by A in $x$ days,

And $\qquad \frac{1}{13}x =$ part done by B in $x$ days;

Confequently $\frac{1}{10}x + \frac{1}{13}x = 1$;

That is $\frac{13}{10}x + x = 13$, or $13x + 10x = 130$;

Or $23x = 130$, and $x = \frac{130}{23} = 5\frac{15}{23}$ days, the time required.

11. If

11. If one agent A, alone, can produce an effect $e$, in the time $a$; and another agent B, alone, in the time $b$; in what time will they both together produce the fame effect?

Let the time fought be denoted by $x$.

Then $a : e :: x : \dfrac{ex}{a} =$ part of the effect produced by A,

And $b : e :: x : \dfrac{ex}{b} =$ part of the effect produced by B,

Hence $\dfrac{ex}{a} + \dfrac{ex}{b} = e$ by the queftion;

Or $\dfrac{x}{a} + \dfrac{x}{b} = 1$;

Theref. $x + \dfrac{ax}{b} = a$;

And $bx + ax = ab$;

Confeq. $x = \dfrac{ab}{a+b} =$ time required.

### QUESTIONS FOR PRACTICE.

1. WHAT two numbers are thofe whofe difference is 7, and fum 33?                           Anf. 13 and 20.

2. To divide the number 75 into two fuch parts, that 3 times the greater may exceed 7 times the lefs by 15.
Anf. 54 and 21.

3. In a mixture of wine and cyder, $\frac{1}{2}$ of the whole *plus* 25 gallons was wine, and $\frac{1}{3}$ part *minus* 5 gallons was cyder: how many gallons were there of each?
Anf. 85 of wine, and 35 of cyder.

4. A bill of 120l. was paid in guineas and moidores, and the number of pieces of both forts that were ufed was juft 100; how many were there of each?       Anf. 50 of each.

5. Two travellers fet out at the fame time from London and York, whofe diftance is 150 miles; one of them goes 8 miles a day, and the other 7; in what time will they meet?
Anf. in 10 days.

6. At a certain election 375 perfons voted for two candidates, and the candidate chofen had a majority of 91; how many voted for each?
Anf. 233 for one, and 142 for the other.

7. What number is that from which, if 5 be fubtracted, $\frac{2}{3}$ of the remainder will be 40?                Anf. 65.

8. A

8. A poft is $\frac{1}{4}$ in the mud, $\frac{1}{3}$ in the water, and 10 feet above the water ; what is the whole length ?    Anf. 24 feet.

9. There is a fiſh whofe tail weighs 9lb. his head weighs as much as his tail and half his body, and his body weighs as much as his head and tail ; what is the whole weight of the fiſh ?                                         Anf. 72lb.

10. After paying away $\frac{1}{4}$ and $\frac{1}{5}$ of my money, I had 66 guineas left in my purfe ; what was in it at firſt ?
                                                        Anf. 120 guineas.

11. A's age is double of B's, and B's is triple of C's, and the fum of all their ages is 140 ; what is the age of each ?
                    Anf. A's $=$ 84, B's $=$ 42, and C's $=$ 14.

12. Two perfons, A and B, lay out equal fums of money in trade ; A gains 126l. and B lofes 87l. and A's money is now double of B's ; what did each lay out ?    Anf. 300l.

13. A perfon bought a chaife, horfe, and harnefs, for 60l. the horfe came to twice the price of the harnefs, and the chaife to twice the price of the horfe and harnefs : what did he give for each ?
                    Anf. 13l. 6s. 8d. for the horfe, 6l. 13s. 4d. for the
                        harnefs, and 40l. for the chaife.

14. Two perfons, A and B, have both the fame income ; A faves $\frac{1}{5}$ of his yearly, but B, by fpending 50l. *per annum* more than A, at the end of 4 years finds himfelf 100l. in debt ; what is their income ?                    Anf. 125l.

15. A perfon has two horfes, and a faddle worth 50l. now if the faddle be put on the back of the firſt horfe, it will make his value double that of the fecond ; but if it be put on the back of the fecond, it will make his value triple that of the firſt ; what is the value of each horfe ?
                    Anf. One 30l. and the other 40l.

16. To divide the number 36 into three fuch parts, that $\frac{1}{2}$ of the firſt, $\frac{1}{3}$ of the fecond, and $\frac{1}{4}$ of the third may be all equal to each other ?   Anf. The parts are 8, 12, and 16.

17. A footman agreed to ferve his mafter for 8l. a year and a livery ; but was turned away at the end of 7 months, and received only 2l. 13s. 4d. and his livery ; what was its value ?                    Anf. 4l. 16s.

18. A perfon was defirous of giving 3d. a-piece to fome beggars ; but found that he had not money enough in his pocket by 8d. ; he therefore gave them each 2d, and had then 3d. remaining ; required the number of beggars ?
                                                        Anf. 11.

                                                        19. A

19. A hare is 50 leaps before a greyhound, and takes 4 leaps to the greyhound's 3 ; but two of the greyhound's leaps are as much as three of the hare's : how many leaps muſt the greyhound take to catch the hare ?               Anſ. 300.

20. A perſon in play loſt $\frac{1}{4}$ of his money, and then won 3 ſhillings ; after which he loſt $\frac{1}{3}$ of what he then had, and then won 2 ſhillings ; laſtly he loſt $\frac{1}{7}$ of what he then had ; and, this done, found he had but 12s. remaining ; what had he at firſt ?               Anſ. 20s.

21. To divide the number 90 into 4 ſuch parts, that if the firſt be increaſed by 2, the ſecond diminiſhed by 2, the third multiplied by 2, and the fourth divided by 2 ; the ſum, difference, product, and quotient ſhall be all equal to each other.   Anſ. The parts are 18, 22, 10, and 40, reſpectively.

22. The hour and minute hand of a clock are exactly together at 12 o'clock : when are they next together ?
               Anſ. 1 hour, $5\frac{5}{11}$ minutes.

23. There is an iſland 73 miles in circumference, and three footmen all ſtart together to travel the ſame way about it : A goes 5 miles a day, B 8, and C 10 : when will they all come together again ?               Anſ. 73 days.

24. How much foreign brandy at 8s. per gallon, and Britiſh ſpirits at 3s. per gallon, muſt be mixed together, ſo that in ſelling the compound at 9s. per gallon, the diſtiller may clear 30 per cent. ?
               Anſ. 51 gallons of brandy, and 14 of ſpirits.

25. A man and his wife uſually drank out a caſk of beer in 12 days ; but when the man was from home, it laſted the woman 30 days ; how many days would the man alone be in drinking it ?               Anſ. 20 days.

26. If A and B together can perform a piece of work in 8 days ; A and C together in 9 days ; and B and C in 10 days ; how many days will it take each perſon to perform the ſame work alone ?
               Anſ. A $14\frac{34}{49}$ days, B $17\frac{23}{41}$, and C $23\frac{7}{31}$.

27. If three agents, A, B, and C, can produce the effects $a$, $b$, $c$, in the times $e$, $f$, $g$, reſpectively ; in what time would they jointly produce the effect $d$ ?

$$\text{Anſ. } d \div \left( \frac{a}{e} + \frac{b}{f} + \frac{c}{g} \right) \text{ time.}$$

QUA.

## QUADRATIC EQUATIONS.

A SIMPLE Quadratic Equation, is that which involves the fquare of the unknown quantity only.

An Affected Quadratic Equation, is that which involves the fquare of the unknown quantity in one term, and the firſt power in another term.

Thus, $ax^2 = b$, is a fimple quadratic equation;
And $ax^2 + bx = c$, is an affected quadratic equation.

The rule for a fimple quadratic equation has been given already.

All affected quadratic equations fall under the three following forms:

$$1. \quad x^2 + ax = b$$
$$2. \quad x^2 - ax = b$$
$$3. \quad x^2 - ax = -b.$$

The rule for finding the value of $x$, in each of thefe equations, is as follows:

### R U L E *.

1. TRANSPOSE all the terms which involve the unknown quantity to one fide of the equation, and the known terms to the other, and let them be ranged according to their dimenſions, as in the forms above.

2. When

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* The fquare root of any quantity may be either + or —, and therefore all quadratic equations admit of two folutions. Thus the fquare root of $+ n^2$ is either $+ n$ or $- n$; for $+ n \times + n$ and $- n \times - n$ are each equal to $+ n^2$. But the fquare root of $- n^2$, or $\sqrt{- n^2}$, is imaginary or impoffible, as neither $+ n$ nor $- n$, when fquared, give $- n^2$.

So, in the firſt form, $x^2 + ax = b$, where $x + \frac{1}{2}a$ is found $= \sqrt{b + \frac{1}{4}a^2}$, the root may be either $+ \sqrt{b + \frac{1}{4}a^2}$, or $-\sqrt{b + \frac{1}{4}a^2}$, fince either of them being multiplied by itfelf will produce $b + \frac{1}{4}a^2$.

And

2. When the fquare of the unknown quantity has any co-efficient prefixed to it, let all the reft of the terms be divided by that co-efficient; which brings the equation to one of the three forms above.

3. Then complete the unknown fide to a fquare in this manner, viz. Take half the co-efficient of the fecond term and fquare it, which fquare add to both fides of the equation, then that fide which involves the unknown quantity will be a complete fquare.

4. Extract the fquare root of both fides of the equation, and the value of the unknown quantity will be determined, as was required, making the root of the known fide either $+$ or $-$, which will give two roots of the equation, or two values of the unknown quantity.

*Note,*

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And this ambiguity is expreffed by writing the uncertain fign $\pm$ before $\sqrt{b + \frac{1}{4}a^2}$; thus $x = \pm\sqrt{b + \frac{1}{4}a^2} - \frac{1}{2}a$.

In this form, where $x = \pm\sqrt{b + \frac{1}{4}a^2} - \frac{1}{2}a$, the firft value of $x$, viz. $x = +\sqrt{b + \frac{1}{4}a^2} - \frac{1}{2}a$, is always affirmative; for fince $\frac{1}{4}a^2 + b$ is greater than $\frac{1}{4}a^2$, the greateft fquare muft necef-farily have the greateft fquare root; therefore $\sqrt{b + \frac{1}{4}a^2}$ will always be greater than $\sqrt{\frac{1}{4}a^2}$, or its equal $\frac{1}{2}a$; and confequently $+\sqrt{b + \frac{1}{4}a^2} - \frac{1}{2}a$ will always be affirmative.

The fecond value, viz. $x = -\sqrt{b + \frac{1}{4}a^2} - \frac{1}{2}a$ will always be negative, becaufe it is compofed of two negative terms. There-fore when $x^2 + ax = b$, we fhall have $x = +\sqrt{b + \frac{1}{4}a^2} - \frac{1}{2}a$ for the affirmative value of $x$, and $x = -\sqrt{b + \frac{1}{4}a^2} - \frac{1}{2}a$ for the negative value of $x$.

In the fecond form, where $x = \pm\sqrt{b + \frac{1}{4}a^2} + \frac{1}{2}a$ the firft value, viz. $x = +\sqrt{b + \frac{1}{4}a^2} + \frac{1}{2}a$ is always affirmative, fince it is compofed of two affirmative terms. The fecond value, viz. $x = -\sqrt{b + \frac{1}{4}a^2} + \frac{1}{2}a$, will always be negative; for fince $b + \frac{1}{4}a^2$ is greater than $\frac{1}{4}a^2$, $\sqrt{b + \frac{1}{4}a^2}$ will be greater than $\sqrt{\frac{1}{4}a^2}$, or its equal $\frac{1}{2}a$; and confequently $-\sqrt{b + \frac{1}{4}a^2} + \frac{1}{2}a$ is always a negative quantity.

Therefore, when $x^2 - ax = b$, we fhall have $x = +\sqrt{b + \frac{1}{4}a^2} + \frac{1}{2}a$ for the affirmative value of $x$, and $x = -\sqrt{b + \frac{1}{4}a^2} + \frac{1}{2}a$ for the negative value of $x$; fo that in both the firft and fecond

forms,

*Note*, 1. The fquare root of the firft fide of the equation is always equal to the unknown quantity, with half the coefficient of the fecond term fubjoined to it.

2. All equations, in which there are two terms involving the unknown quantity, and which have the index of the one juft double that of the other, are refolved like quadratics, by completing the fquare, as above.

Thus, $x^4 + ax^2 = b$, or $x^{2n} + ax^n = b$, or $x + ax^{\frac{1}{2}} = b$, are the fame as quadratics, and the value of the unknown quantity may be determined accordingly.

### EXAMPLES.

1. Given $x^2 + 4x = 140$; to find $x$.

Firft, $x^2 + 4x + 4 = 140 + 4 = 144$, by completing the fquare.

Then $\sqrt{x^2 + 4x + 4} = \sqrt{144}$, by extracting the roots;
Or, which is the fame thing, $x + 2 = \pm 12$.
Theref. $x = \pm 12 - 2 = 10$ or $- 14$, the two roots.

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forms. the unknown quantity has always two values, one of which is pofitive, and the other negative.

In the third form, where $x = \pm \sqrt{\frac{1}{4}a^2 - b} + \frac{1}{2}a$, both the values of $x$ will be pofitive, fuppofing $\frac{1}{4}a^2$ is greater than $b$. For the firft value, viz. $x = + \sqrt{\frac{1}{4}a^2 - b} + \frac{1}{2}a$ will then be affirmative, being compofed of two affirmative terms.

The fecond value, viz. $x = - \sqrt{\frac{1}{4}a^2 - b} + \frac{1}{2}a$ is affirmative alfo; for fince $\frac{1}{4}a^2$ is greater than $\frac{1}{4}a^2 - b$, $\sqrt{\frac{1}{4}a^2}$ or $\frac{1}{2}a$ is greater than $\sqrt{\frac{1}{4}a^2 - b}$; and confequently $- \sqrt{\frac{1}{4}a^2 - b} + \frac{1}{2}a$ will always be an affirmative quantity. Therefore, when $x^2 - ax = - b$, we fhall have $x = + \sqrt{\frac{1}{4}a^2 - b} + \frac{1}{2}a$, and alfo $x = - \sqrt{\frac{1}{4}a^2 - b} + \frac{1}{2}a$ for the values of $x$, both affirmative.

But in this third form, if $b$ be greater than $\frac{1}{4}a^2$, the folution of the propofed queftion will be impoffible. For fince the fquare of any quantity (whether that quantity be affirmative or negative) is always affirmative, the fquare root of a negative quantity is impoffible, and cannot be affigned. But if $b$ be greater than $\frac{1}{4}a^2$, then $\frac{1}{4}a^2 - b$ is a negative quantity; and therefore $\sqrt{\frac{1}{4}a^2 - b}$ is impoffible. or imaginary; confequently, in that cafe, $x = \frac{1}{2}a \pm \sqrt{\frac{1}{4}a^2 - b}$, or the two roots or values of $x$, are both impoffible, or imaginary.

2. Given

2. Given $x^2 - 6x + 8 = 80$; to find $x$.

First, $x^2 - 6x = 80 - 8 = 72$, by transposition;
Then $x^2 - 6x + 9 = 72 + 9 = 81$, by completing the sq.
And $x - 3 = \sqrt{81} = \pm 9$, by extracting the root;
Theref. $x = \pm 9 + 3 = 12$ or $- 6$.

3. Given $2x^2 + 8x - 20 = 70$; to find $x$.

First, $2x^2 + 8x = 70 + 20 = 90$ by transposition;
Then $x^2 + 4x = 45$ by dividing by 2;
And $x^2 + 4x + 4 = 49$ by completing the square;
Hence $x + 2 = \sqrt{49} = \pm 7$ by extracting the root;
Consequently $x = \pm 7 - 2 = 5$ or $- 9$.

4. Given $3x^2 - 3x + 6 = 5\frac{1}{3}$; to find $x$.

Here, $x^2 - x + 2 = 1\frac{7}{9}$ by dividing by 3,
And $x^2 - x = 1\frac{7}{9} - 2$ by transposition:
Also $x^2 - x + \frac{1}{4} = 1\frac{7}{9} - 2 + \frac{1}{4} = \frac{1}{36}$ by compl. the sq.
Hence $x - \frac{1}{2} = \sqrt{\frac{1}{36}} = \pm \frac{1}{6}$ by evolution;
Therefore $x = \pm \frac{1}{6} + \frac{1}{2} = \frac{2}{3}$ or $\frac{1}{3}$.

5. Given $\frac{1}{2}x^2 - \frac{1}{3}x + 20\frac{1}{2} = 42\frac{2}{3}$; to find $x$.

Here, $\frac{1}{2}x^2 - \frac{1}{3}x = 42\frac{2}{3} - 20\frac{1}{2} = 22\frac{1}{6}$ by transposition;
And $x^2 - \frac{2}{3}x = 44\frac{1}{3}$ by multiplying by 2.
Then $x^2 - \frac{2}{3}x + \frac{1}{9} = 44\frac{1}{3} + \frac{1}{9} = 44\frac{4}{9}$ by compl. squ.
Hence $x - \frac{1}{3} = \sqrt{44\frac{4}{9}} = \pm 6\frac{2}{3}$ by evolution;
Therefore $x = \pm 6\frac{2}{3} + \frac{1}{3} = 7$ or $- 6\frac{1}{3}$.

6. Given $ax^2 + bx = c$; to find $x$.

First, $x^2 + \frac{b}{a}x = \frac{c}{a}$ by division;

Then $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{c}{a} + \frac{b^2}{4a^2}$ by compl. square.

And $x + \frac{b}{2a} = \sqrt{(\frac{c}{a} + \frac{b^2}{4a^2})} = \pm\sqrt{\frac{4ac + b^2}{4a^2}}$ by evolut.

Therefore $x = \pm\sqrt{(\frac{4ac + b^2}{4a^2})} - \frac{b}{2a}$.

7. Given $ax^2 - bx + c = d$; to find $x$.

Here, $ax^2 - bx = d - c$ by transposition;

And $x^2 - \frac{b}{a}x = \frac{d - c}{a}$ by division;

Alfo $x^2 - \dfrac{b}{a}x + \dfrac{b^2}{4a^2} = \dfrac{d-c}{a} + \dfrac{b^2}{4a^2}$ by compl. the fq.

Hence $x - \dfrac{b}{2a} = \pm\sqrt{(\dfrac{d-c}{a} + \dfrac{b^2}{4a^2})}$ by evolution;

Therefore $x = \dfrac{b}{2a} \pm\sqrt{(\dfrac{d-c}{a} + \dfrac{b^2}{4a^2})}$.

8. Given $x^4 + 2ax^2 = b$; to find $x$.

Here, $x^4 + 2ax^2 + a^2 = b + a^2$ by compl. the fq.

And $x^2 + a = \pm\sqrt{b+a^2}$ by evolution;

Hence $x^2 = \pm\sqrt{b+a^2} - a$, by tranfpofition;

And confequently $x = \pm\sqrt{-a \pm \sqrt{b+a^2}}$.

9. Given $ax^n - bx^{\frac{n}{2}} - c = -d$; to find $x$.

Firft, $ax^n - bx^{\frac{n}{2}} = c - d$ by tranfpofition;

And $x^n - \dfrac{b}{a}x^{\frac{n}{2}} = \dfrac{c-d}{a}$ by divifion;

Alfo $x^n - \dfrac{b}{a}x^{\frac{n}{2}} + \dfrac{b^2}{4a^2} = \dfrac{c-d}{a} + \dfrac{b^2}{4a^2}$ by compl. the fq.

Hence $x^{\frac{n}{2}} - \dfrac{b}{2a} = \pm\sqrt{(\dfrac{c-d}{a} + \dfrac{b^2}{4a^2})}$ by evolution;

Therefore $x^{\frac{n}{2}} = \dfrac{b}{2a} \pm\sqrt{(\dfrac{c-d}{a} + \dfrac{b^2}{4a^2})}$;

And confeq. $x = (\dfrac{b}{2a} \pm\sqrt{\dfrac{4ac - 4ad + b^2}{4a^2}})^{\frac{2}{n}}$.

### EXAMPLES FOR PRACTICE.

1. Given $x^2 - 8x + 10 = 19$; to find $x$.  Anf. $x = 9$.
2. Given $x^2 - x - 40 = 170$; to find $x$.  Anf. $x = 15$.
3. Given $3x^2 + 2x - 9 = 76$; to find $x$.  Anf. $x = 5$.
4. Given $\frac{1}{2}x^2 - \frac{1}{3}x + 7\frac{3}{8} = 8$; to find $x$.  Anf. $x = 1\frac{1}{2}$.
5. Given $2x^4 - x^2 = 496$; to find $x$.  Anf. $x = 4$.
6. Given $\frac{1}{2}x - \frac{1}{3}\sqrt{x} = 22\frac{1}{6}$; to find $x$.  Anf. $x = 49$.
7. Given $\frac{2}{3}x^2 + \frac{3}{4}x = \frac{4}{5}$; to find $x$.  Anf. $x = \cdot6689$.
8. Given $x^6 + 6x^3 = 2$; to find $x$. Anf. $x = \sqrt[3]{-3 \pm \sqrt{11}}$.
9. Given $x^2 + x = a$; to find $x$.  Anf. $x = \sqrt{a + \frac{1}{4}} - \frac{1}{2}$.

10. Given

10. Given $x - \sqrt{x} = a$; to find $x$. Anf. $x = (\frac{1}{2} \pm \sqrt{a + \frac{1}{4}})^2$.

11. Given $3x^{2n} - 2x^n = 25$; to find $x$.

Anf. $x = (\frac{1}{3}\sqrt{76} + \frac{1}{3})^{\frac{1}{n}}$.

12. Given $\sqrt{1 + x} - 2\sqrt[4]{1 + x} = 4$; to find $x$.

Anf. $x = (1 + \sqrt{5})^4 - 1$.

## QUESTIONS PRODUCING QUADRATIC EQUATIONS.

1. To find two numbers whofe difference is 8, and pro-
duct 240.

Let $x = $ to the leaft number.
Then will $x + 8 = $ to the greater,
And $x \times (x + 8) = x^2 + 8x = 240$ by the queftion;
Thence $x^2 + 8x + 16 = 240 + 16 = 256$ by complet-
    ing the fquare;
And hence $x + 4 = \sqrt{256} = 16$ by evolution;
Theref. $x = 16 - 4 = 12$ the lefs number,
And $12 + 8 = 20$ the greater.

2. To divide the number 60 into two fuch parts, that
their product may be 864.

Let $x = $ the greater part,
Then will $60 - x = $ the lefs,
And $x \times (60 - x) = 60x - x^2 = 864$ by the queftion;
That is $x^2 - 60x = -864$;
Then $x^2 - 60x + 900 = -864 + 900 = 36$ by com-
    pleting the fquare;
Hence $x - 30 = \pm\sqrt{36} = \pm 6$ by extracting the root;
Theref. $x = 30 \pm 6 = 36$ or $24$, the two parts.

3. Given the fum of two numbers $= 10 \ (a)$, and the fum
of their fquares $= 58 \ (b)$; to find thofe numbers.

Let $x = $ the greater of the two numbers,
Then will $a - x = $ the lefs;
And $x^2 + (a - x)^2 = 2x^2 + a^2 - 2ax = b$ by the queftion,
Or $x^2 + \frac{1}{2}a^2 - ax = \frac{1}{2}b$ by divifion,
Or $x^2 - ax = \frac{1}{2}b - \frac{1}{2}a^2$ by tranfpofition;
Then $x^2 - ax + \frac{1}{4}a^2 = \frac{1}{2}b - \frac{1}{4}a^2$ by compl. the fq.
Hence $x - \frac{1}{2}a = \pm \frac{1}{2}\sqrt{2b - a^2}$ by extracting the root;
Theref. $x = \frac{1}{2}a \pm \frac{1}{2}\sqrt{2b - a^2}$ the two numbers.

S 2

That

That is, $x = \frac{1}{2}a + \frac{1}{2}\sqrt{2b - a^2}$ the greater number,

And $\quad x = \frac{1}{2}a - \frac{1}{2}\sqrt{2b - a^2}$ the lefs number.

Hence thefe two theorems, being put into numbers, give 7 and 3, for the numbers required.

4. Sold a piece of cloth for 24l. and gained as much *per cent.* as the cloth coft me ; what was the price of the cloth ?

Let $x =$ pounds the cloth coft,
Then $24 - x =$ whole gain,
But $100 : x :: x : 24 - x$ by the queftion,
Or $x^2 = 100 \times \overline{24 - x} = 2400 - 100x$,
That is, $x^2 + 100x = 2400$ ;
Then $\quad x^2 + 100x + 2500 = 4900$ by compl. the fq.
And $x + 50 = \sqrt{4900} = 70$ by extraction of roots,
Confequently $x = 70 - 50 = 20$l. $=$ price of the cloth.

5. A perfon bought a number of oxen for 80l. and if he had bought 4 more for the fame money, he would have paid 1l. lefs for each : how many did he buy ?

Let the number of oxen be reprefented by $x$,

Then will $\dfrac{80}{x}$ be the price of each,

And $\dfrac{80}{x + 4} =$ price of each, if $x + 4$ had coft 80l.

But $\dfrac{80}{x} = \dfrac{80}{x + 4} + 1$ by the queftion,

Or $80 = \dfrac{80x}{x + 4} + x$, by multiplication,

Or $80x + 320 = 80x + x^2 + 4x$, by the fame,
That is, $x^2 + 4x = 320$ ;
Then $x^2 + 4x + 4 = 324$ by completing the fquare,
And $x + 2 = \sqrt{324} = 18$ by evolution,
Confeq. $x = 18 - 2 = 16$, the numb. of oxen required.

6. What two numbers are thofe, whofe fum, product, and difference of their fquares, are all equal to each other ?

Let $x =$ the greater number,
And $y =$ the lefs,

Then $\left\{ \begin{array}{l} x + y = xy \\ x + y = x^2 - y^2 \end{array} \right\}$ by the queftion ;

Hence $1 = \dfrac{x^2 - y^2}{x + y} = x - y$, or $x = y + 1$ from the 2d equation,

$\quad$ I

Alfo

Alfo $\overline{y+1}+y=\overline{y+1}\times y$ from the firft equation,

Or $2y+1=y^2+y$,

That is, $y^2-y=1$;

Then $y^2-y+\frac{1}{4}=1\frac{1}{4}$ by completing the fquare,

Alfo $y-\frac{1}{2}=\sqrt{1\frac{1}{4}}=\sqrt{\frac{5}{4}}=\frac{1}{2}\sqrt{5}$ by evolution,

Confequently $y=\frac{1}{2}\sqrt{5}+\frac{1}{2}$,

And $x=y+1=\frac{1}{2}\sqrt{5}+\frac{3}{2}$.

And if thefe expreffions be turned into numbers, we fhall

have $x=2\cdot6180+$

and $y=1\cdot6180+$

7. There are four numbers in arithmetical progreffion, of which the product of the two extremes is 45, and that of the means 77; what are the numbers?

Let $x=$ the lefs extreme,

and $y=$ the common difference;

Then $x$, $x+y$, $x+2y$, $x+3y$, will be the four numbers,

Hence $x\times\overline{x+3y}=x^2+3xy=45$, $\qquad$ by the

And $\overline{x+y}\times\overline{x+2y}=x^2+3xy+2y^2=77$ $\}$ queftion.

Hence $2y^2=77-45=32$ by fubtraction,

And $y^2=\frac{32}{2}=16$ by divifion,

Or $y=\sqrt{16}=4$ by evolution;

Therefore $x^2+3xy=x^2+12x=45$ by the 1ft equation,

And $x^2+12x+36=45+36=81$ by completing the fq.

Alfo $x+6=\sqrt{81}=9$ by the extraction of roots;

Confeq. $x=9-6=3$,

And the numbers are 3, 7, 11, and 15.

8. To find 3 numbers in geometrical progreffion, whofe fum fhall be 14, and the fum of their fquares 84.

Let $x, y$, and $z$ be the three numbers fought,

Then $xz=y^2$ by the nature of proportion,

And $\begin{cases}x+y+z=14\\x^2+y^2+z^2=84\end{cases}$ by the queftion;

Hence $x+z=14-y$ by the 2d equation,

And $x^2+2xz+z^2=196-28y+y^2$ by fquaring.

Or $x^2+z^2+2y^2=196-28y+y^2$ by putting $2y^2$ for its equal $2xz$,

That is, $x^2+z^2+y^2=196-28y$ by fubtraction,

Or $196-28y=84$ by equality,

Hence $y=\dfrac{196-84}{28}=4$ by tranfpofition and divifion.

Again,

Again, $xz = y^2 = 16$, or $x = \dfrac{16}{z}$ by the 1ft equation,

And $x + y + z = \dfrac{16}{z} + 4 + z = 14$ by the 2d equation,

Or $16 + 4z + z^2 = 14z$, or $z^2 - 10z = -16$;
Then $z^2 - 10z + 25 = 25 - 16 = 9$ by compl. fq.
And $z - 5 = \pm\sqrt{9} = \pm 3$ by extracting the roots;
Hence $z = 5 \pm 3 = 8$ and 2, the two other numbers,
That is, $x = 2$, and $z = 8$,
And the three numbers are 2, 4, 8.

### QUESTIONS FOR PRACTICE.

1. WHAT two numbers are thofe, whofe fum is 20, and their product 36 ?　　　　　　　　　Anf. 2 and 18.

2. To divide the number 60 into two fuch parts, that their product may be to the fum of their fquares, in the ratio of 2 to 5.　　　　　　　　　Anf. 20 and 40.

3. The difference of two numbers is 3, and the difference of their cubes is 117; what are thofe numbers?

　　　　　　　　　　　　　　　　Anf. 2 and 5.

4. A company at a tavern had 8l. 15s. to pay for their reckoning; but, before the bill was fettled, two of them left the room, and then thofe who remained had 10s. a-piece more to pay than before: how many were there in company?　　　　　　　　　Anf. 7.

5. A grazier bought as many fheep as coft him 60l. and, after referving 15 out of the number, he fold the remainder for 54l. and gained 2s. a head by them; how many fheep did he buy?　　　　　　　　　Anf. 75.

6. There are two numbers whofe difference is 15, and half their product is equal to the cube of the leffer number; what are thofe numbers?　　　　　　　　　Anf. 3 and 18.

7. A perfon bought cloth for 33l. 15s. which he fold again at 2l. 8s. per piece, and gained by the bargain as much as one piece coft him; required the number of pieces?

　　　　　　　　　　　　　　　　Anf. 15.

8. What number is that, which, when divided by the product of its two digits, the quotient is 3; and if 18 be added to it, the digits will be inverted?　　　Anf. 24.

9. What two numbers are thofe, whofe fum multiplied by the greater is equal to 77; and whofe difference multiplied by the leffer is equal to 12?　　　Anf. 4 and 7.

10. To

10. To find a number fuch, that if you fubtract it from 10, and multiply the remainder by the number itfelf, the product fhall be 21. Anf. 7 or 3.

11. To divide 100 into two fuch parts, that the fum of their fquare roots may be 14. Anf. 64 and 36.

12. It is required to divide the number 24 into two fuch parts, that their product may be equal to 35 times their difference. Anf. 10 and 14.

13. The fum of two numbers is 8, and the fum of their cubes is 152; what are the numbers? Anf. 3 and 5.

14. The fum of two numbers is 7, and the fum of their 4th powers is 641; what are the numbers? Anf. 2 and 5.

15. The fum of two numbers is 6, and the fum of their 5th powers is 1056; what are the numbers? Anf. 2 and 4.

16. The fum of four numbers in arithmetical progreffion is 56, and the fum of their fquares is 864; what are the numbers? Anf. 8, 12, 16, and 20.

17. To find four numbers in geometrical progreffion, whofe fum is 15, and the fum of their fquares 85?

Anf. 1, 2, 4, and 8.

18. It is required to find four numbers in arithmetical progreffion, fuch that their common difference may be 4, and their continued product 176985.

Anf. 15, 19, 23, and 27.

19. Two partners, A and B, gained 140l. by trade; A's money was 3 months in trade, and his gain was 60l. lefs than his ftock; and B's money, which was 50l. more than A's, was in trade 5 months; what was A's ftock?

Anf. 100l.

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## RESOLUTION OF CUBIC AND HIGHER EQUATIONS.

A CUBIC Equation, or Equation of the 3d degree or power, is one that contains the third power of the unknown quantity. As $x^3 - ax^2 + bx = c$.

A *Biquadratic*, or Double Quadratic, is an equation that contains the 4th power of the unknown quantity. As
$x^4 - ax^3 + bx^2 - cx = d$.

An

An Equation of the 5th Power, or Degree, is one that contains the 5th power of the unknown quantity. As
$$x^5 - ax^4 + bx^3 - cx^2 + dx = e.$$

An Equation of the 6th Power or Degree, is one that contains the 6th power of the unknown quantity. As
$$x^6 - ax^5 + bx^4 - cx^3 + dx^2 - ex = f.$$

And so on, for all other higher powers. Where it is to be noted however, that all the powers or terms, in the equation, are suppofed to be freed from furds or fractional exponents.

There are many particular, and prolix rules, ufually given for the refolution of fome of the above-mentioned powers or equations. But they may be all eafily refolved by the following eafy rule of Double Pofition, fometimes called Trial-and-Error.

### R U L E.

1. Find, by trial, two numbers, as near the true root as you can, and fubftitute them feparately in the given equation, inftead of the unknown quantity ; marking the errors which arife from each of them.

2. Multiply the difference of the two numbers, found or taken by trial, by the leaft error, and divide the product by the difference of the errors, when they are alike, but by their fum when they are unlike. Or fay, As the difference or fum of the errors is to the difference of the two numbers, fo is the leaft error to the correction of its fuppofed number.

3. Add the quotient, laft found, to the number belonging to the leaft error, when that number is too little, but fubtract it when too great, and the refult will give the true root *nearly*.

4. Take this root and the neareft of the two former, or any other that may be found nearer ; and, by proceeding in like manner as above, a root will be had ftill nearer than before ; and fo on to any degree of exactnefs required.

*Note* 1. It is beft to employ always two affumed numbers that fhall differ from each other only by unity in the laft figure on the right hand ; becaufe then the difference, or multiplier, is only 1.

*Note* 2. The above rule is the 2d rule for Double Pofition, given in the Arithmetic, pag. 140, (where it is demonftrated). It is the eafieft to be ufed here, and each new operation commonly doubles the number of true figures in the root.

### EXAMPLES.

Ex. 1. To find the root of the cubic equation $x^3 + x^2 + x = 100$, or the value of $x$ in it.

Here it is foon found that $x$ lies between 4 and 5. Affume therefore thefe two numbers, and the operation will be as follows :

| 1ft. Sup. | | | | 2d Sup. |
|---|---|---|---|---|
| 4 | - - | $x$ | - - | 5 |
| 16 | - - | $x^2$ | - - | 25 |
| 64 | - - | $x^3$ | - - | 125 |
| .84 | - | fums | - | 155 |
| —16 | - | errors | - | +55 |

the fum of which is 71.
Then as $71 : 1 :: 16 : \cdot225$.
Hence $x = 4\cdot225$ nearly.

Again, fuppofe 4·2 and 4·3, and repeat the work as follows :

| 1ft. Sup. | | | | 2d Sup. |
|---|---|---|---|---|
| 4·2 | - | $x$ | - | 4·3 |
| 17 64 | - | $x^2$ | - | 18·49 |
| 74·088 | - | $x^3$ | - | 79·507 |
| 95·928 | - | fums | - | 102·297 |
| —4.072 | - | errors | - | +2·297 |

the fum of which is 6·369.
As $6\cdot369 : 1 :: 2\cdot297 : 0\cdot036$
This taken from - 4·300

leaves $x$ nearly $= 4\cdot264$

Again, fuppofe 4·264, and 4·265, and work as follows :

| 4·264 | - | $x$ | - | 4·265 |
|---|---|---|---|---|
| 18·181696 | - | $x^2$ | - | 18·190225 |
| 77·526752 | - | $x^3$ | - | 77·581310 |
| 99·972448 | - | fums | - | 100·036535 |
| —0·027552 | - | errors | - | +0·036535 |

the fum of which is ·064087.

Then as $\cdot064087 : \cdot001 :: 4\cdot264 : 0\cdot0004299$
To this adding - 4·264

gives $x$ very nearly $= 4\cdot2644299$

The work of the example above might have been much fhortened, by the ufe of the table of powers at p. 90, &c, which would have given two or three figures by infpection. But the example has been worked out fo particularly as it is, the better to fhew the method.

Ex. 2. To find the root of the equation $x^3 - 15x^2 + 63x = 50$, or the value of $x$ in it.

Here it foon appears that $x$ is very little above 1.

Suppofe therefore 1·0 and 1·1, and work as follows :

| 1·0 | - | $x$ | - | 1·1 |

| .63·0 | - | 63$x$ | - | 69·3 |
| —15 | | —15$x^2$ | - | —18·15 |
| 1 | - | $x^3$ | - | 1·1 |

| 49 | - | fums | - | 52·25 |

—1　-　errors　-　+2·25
3·25 fum of the errors.
As 3·25 : 1 :: 1 : ·03 correct.
　　　　　　1·00

Hence $x = 1$·03 nearly.

Again, fuppofe the two numbers 1·03 and 1·02, &c, as follows :

| 1·03 | - | $x$ | - | 1·02 |

| 64·89 | - | 63$x$ | 64·26 |
| —15·9135 | —15$x^2$ | —15·6060 |
| 1·092727 | $x^3$ | 1·061208 |

50·069227 fums 49·715208

+·069227 errors—·284792
·284792

As ·354019 : ·01 :: ·069227 : ·0019555
This taken from　1·03

leaves $x$ nearly = 1·02804

*Note* 3. Every equation has as many roots as it contains dimenfions, or as there are units in the index of its higheft power. That is, a fimple equation has only one value of the root; but a quadratic equation has two values or roots, a cubic equation has three roots, a biquadratic equation has four roots, and fo on.

And when one of the roots of an equation has been found by approximation, as above, the reft may be found as follows. —Take for a dividend, the given equation, with the known term tranfpofed, with its fign changed, to the unknown fide of the equation ; and for a divifor, take $x$ minus the root juft found. Divide the faid dividend by the divifor, and the quotient will be the equation depreffed a degree lower than the given one.

Find a root of this new equation by approximation, as before, and it will be a fecond root of the original equation. Then, by means of this root, deprefs the fecond equation one degree lower, and from thence find a third root, and fo on, till the equation be reduced to a quadratic ; then the two roots of this being found, by the method of completing the fquare, they will make up the remainder of the roots. Thus, in the foregoing equation, having found one root to be 1·02804, connect it by minus with $x$ for a divifor, and the equation for a dividend, &c, as follows :

$x$ —

$x - 1.02804) x^3 - 15x^2 + 63x - 50 (x^2 - 13.97196x +$
$$48.63627 = 0.$$

Then the two roots of this quadratic equation, or - - -
$x^2 - 13.97196x = - 48.63627$, by completing the fquare,
are 6.57653 and 7.39543, which are alfo the other two
roots of the given cubic equation. So that all the three
roots of that equation, viz. $x^3 - 15x^2 + 63x = 50$,

are 1.02804,⎤ and the fum of all the roots is found to
and 6.57653, ⎪ be 15, being equal to the co-efficient of
and 7.39543, ⎬ the 2d term of the equation, which the
⎪ fum of the roots always ought to be, when
fum 15.00000 ⎦ they are right.

*Note* 4. It is alfo a particular advantage of the foregoing
rule, that it is not neceffary to prepare the equation, as for
other rules, by reducing it to the ufual final form and ftate
of equations. Becaufe the rule may be applied at once to an
unreduced equation, though it be ever fo much embarraffed
by furd and compound quantities. As in the following
example :

Ex. 3. Let it be required to find the root $x$ of the equation
$$\sqrt{144x^2 - (x^2 + 20)^2} + \sqrt{196x^2 - (x^2 + 24)^2} = 114,$$
or the value of $x$ in it.

By a few trials, it is foon found that the value of $x$ is but
little above 7. Suppofe, therefore, firft, that $x$ is $= 7$, and
then $x = 8$.

First, when $x = 7$,                    Second, when $x = 8$.

| | | |
|---|---|---|
| 47.906 | - $\sqrt{144x^2 - (x^2 + 20)^2}$ - | 46.476 |
| 65.384 | - $\sqrt{196x^2 - (x^2 + 24)^2}$ - | 69.283 |
| 113.290 | - the fums of thefe - | 115.759 |
| 114.000 | - the true number - | 114.000 |
| —0.710 | - the two errors - | +1.759 |
| +1.759 | | |

As      2.469 : 1 :: 0.710 : 0.2 nearly
7.0
$$x = 7.2 \text{ nearly}$$

Suppofe again $x = 7\cdot2$, and then, becaufe it turns out too great, fuppofe $x$ alfo $= 7\cdot1$, &c, as follows :

| Supp. $x = 7\cdot2$. | | Supp. $x = 7\cdot1$ |
|---|---|---|
| $47\cdot990$ - | $\sqrt{144x^2 - (x^2 + 20)^2}$ | - $47\cdot973$ |
| $66\cdot402$ - | $\sqrt{196x^2 - (x^2 + 20)^2}$ | - $65\cdot904$ |
| $114\cdot392$ - | the fums of thefe - | - $113\cdot877$ |
| $114\cdot000$ - | the true number - | - $114\cdot000$ |
| $+0\cdot392$ - | the errors - | - $-0\cdot123$ |
| $0\cdot123$ - | | |

$\cdot515 : \cdot123 :: \cdot1 : \cdot024$ the correction,
$$7\cdot100$$

Therefore $x = 7\cdot124$ nearly the root required.

*Note* 5. The fame rule alfo, among other more difficult forms of equations, fucceeds very well in what are called exponential ones, or thofe which have an unknown quantity in the exponent of the power; as in the following example:

Ex. 4. To find the value of $x$ in the exponential equation $x^x = 100$.

For more eafily refolving fuch kind of equations, it is convenient to take the logarithms of them, and then compute the terms by means of a table of logarithms. Thus, the logarithms of the two fides of the prefent equation are, $x \times$ log. of $x = 2$ the log. of 100. Then, by a few trials it is foon perceived that the value of $x$ is fomewhere between the two numbers 3 and 4, and indeed nearly in the middle between them, but rather nearer the latter than the former. Taking therefore firft $x = 3\cdot5$, and then $= 3\cdot6$, and working with the logarithms, the operation will be as follows:

| Firft Supp. $x = 3\cdot5$. | Second Supp. $x = 3\cdot6$. |
|---|---|
| Log. of $3\cdot5 = 0\cdot5440680$ | Log. of $3\cdot6 = 0\cdot5563025$ |
| then $3\cdot5 \times$ log. $3\cdot5 = 1\cdot904238$ | then $3\cdot6 \times$ log. $3\cdot6 = 2\cdot002689$ |
| the true number $2\cdot000000$ | the true number $2\cdot000000$ |
| error, too little, $-\cdot095762$ | error, too great, $+\cdot002689$ |
| $002689$ | |

$\cdot098451$ fum of the errors. Then,

As

As ·098451 : ·1 :: ·002689 : 0·00273 the correction
taken from 3·60000

leaves - 3·59727 = $x$ nearly.

On trial, this is found to be a very fmall matter too little.
Take therefore again, $x$ = 3·59727, and next = 3·59728,
and repeat the operation as follows:

| Firft, Supp. $x$ = 3.59727. | Second, Supp. $x$ = 3·59728. |
|---|---|
| Log. of 3·59727 is 0.5559731 | Log. of 3·59728 is 0·5559743 |
| 3·59727 × log. | 3·59728 × log. |
| of 3·59727 = 1·9999854 | of 3·59728 = 1·9999953 |
| the true number 2·0000000 | the true number 2·0000000 |
| error, too little, —0·0000146 | error, too little, —0·0000047 |
| —0·0000047 | |

0·0000099 diff. of the errors.    Then,

As ·0000099 : ·00001 :: ·0000047 : 0·00000474747 the cor.
added to    -    3·59728000000

gives nearly the value of $x$ = 3·59728474747

Ex. 5. To find the value of $x$ in the equation $x^3 + 10x^2 + 5x = 2600$.    Anf. $x$ = 11·00673.

Ex. 6. To find the value of $x$ in the equation $x^3 - 2x = 5$.    Anf. 2·004551.

Ex. 7. To find the value of $x$ in the equation $x^3 + 2x^2 - 23x = 70$.    Anf. $x$ = 5·1349.

Ex. 8. To find the value of $x$ in the equation $x^3 - 17x^2 + 54x = 350$.    Anf. $x$ = 14·95407.

Ex. 9. To find the value of $x$ in the equation $x^4 - 3x^2 - 75x = 10000$.    Anf. $x$ = 10·2615.

Ex. 10. To find the value of $x$ in the equation $2x^4 - 16x^3 + 40x^2 - 30x = -1$.    Anf. $x$ = 1·284724.

Ex. 11. To find the value of $x$ in the equation $x^5 + 2x^4 + 3x^3 + 4x^2 + 5x = 54321$.    Anf. $x$ = 8·414455.

Ex. 12. To find the value of $x$ in the equation $x^x = 123456789$.    Anf. $x$ = 8·6400268.

## OF SIMPLE INTEREST.

As the intereſt of any ſum, for any time, is directly pro-
portional to the principal ſum, and to the time ; therefore
the intereſt of 1 pound, for 1 year, being multiplied by any
given principal ſum, and by the time of its forbearance, in
years and parts, will give its intereſt for that time. That is,
if there be put

$r =$ the rate of intereſt of 1 pound per annum,
$p =$ any principal ſum lent,
$t =$ the time it is lent for, and
$a =$ the amount, or ſum of principal and intereſt; then
is $prt =$ the intereſt of the ſum $p$, for the time $t$, and con-
ſeq. $p + prt$ or $p \times (1 + rt) = a$, the amount for that time.

From this expreſſion, other theorems can eaſily be de-
duced, for finding any of the quantities above mentioned:
which theorems, collected altogether, will be as below ;

1ſt, $a = p + prt$, the amount,

2d, $p = \dfrac{a}{1 + rt}$, the principal,

3d, $r = \dfrac{a - p}{pt}$, the rate,

4th, $t = \dfrac{a - p}{pr}$, the time.

*For Example.* Let it be required to find, in what time
any principal ſum will double itſelf, at any rate of ſimple
intereſt.

In this caſe, we muſt uſe the firſt theorem, $a = p + prt$,
in which the amount $a$ muſt be made $= 2p$, or double the
principal, that is, $p + prt = 2p$, or $prt = p$, or $rt = 1$ ;
and hence $t = \dfrac{1}{r}$.

Here, $r$ being the intereſt of 1l. for 1 year, it follows,
that the doubling at ſimple intereſt, is equal to the quotient
of any ſum divided by its intereſt for 1 year. So, if the
rate of intereſt be 5 per cent. then $100 \div 5 = 20$, is the
time of doubling at that rate.

Or, the 4th theorem gives at once

$$t = \frac{a - p}{pr} = \frac{2p - p}{pr} = \frac{2 - 1}{r} = \frac{1}{r}, \text{ the ſame as before.}$$

COM-

## COMPOUND INTEREST.

BESIDE the quantities concerned in Simple Interest, namely,

$p =$ the principal fum,
$r =$ the rate, or intereft of 1l. for 1 year,
$a =$ the whole amount of the principal and intereft,
$t =$ the time,

there is another quantity employed in Compound Intereft, viz. the ratio of the rate of intereft, which is the amount of 1l. for 1 time of payment, and which here let be denoted by R, viz.

$R = 1 + r$, the amount of 1l. for 1 time.

Then, the particular amounts for the feveral times may be thus computed, viz. As 1l. is to its amount for any time, fo is any propofed principal fum, to its amount for the fame time; that is, as

$1l : R :: p \quad : pR$, the 1ft year's amount,
$1l : R :: pR \quad : pR^2$, the 2d year's amount,
$1l : R :: pR^2 : pR^3$, the 3d year's amount,
and fo on.

Therefore, in general, $pR^t = a$ is the amount for the $t$ year, or $t$ time of payment. From whence the following general theorems are deduced :

1ft, $a = pR^t$, the amount.

2d, $p = \dfrac{a}{R^t}$, the principal,

3d, $R = \sqrt[t]{\dfrac{a}{p}}$, the ratio,

4th, $t = \dfrac{\text{log. of } a - \text{log of } p}{\text{log of R}}$, the time.

From which, any one of the quantities may be found, when the reft are given.

As to the whole intereft, it is found by barely fubtracting the principal $p$ from the amount $a$.

*Example.* Suppofe it be required to find, in how many years any principal fum will double itfelf, at any propofed rate of compound intereft.

In this cafe the 4th theorem muft be employed, making $a = 2p$; and then it is

$$t = \frac{l.\, a - l.\, p}{\log.\, R} = \frac{l.\, 2p - l.\, p.}{\log.\, R} = \frac{\log.\, 2}{\log.\, R}.$$

So, if the rate of intereft be 5 per cent. per annum; then $R = 1 + \cdot 05 = 1 \cdot 05$; and hence

$$t = \frac{\log.\, 2}{\log.\, 1 \cdot 05} = \frac{\cdot 3010300}{\cdot 0211893} = 14 \cdot 2067 \text{ nearly};$$

that is, any fum doubles itfelf in $14\frac{1}{5}$ years nearly, at the rate of 5 per cent. per annum compound intereft.

From hence, and from the like quéftion in Simple Intereft, above given, are deduced the times in which any fum doubles itfelf, at feveral rates of intereft, both fimple and compound; viz.

| At | | At Simp. Int. | At Comp. Int. |
|---|---|---|---|
| | | Years. | Years. |
| 2 | | 50 | 35.0028 |
| $2\frac{1}{2}$ | | 40 | 28·0701 |
| 3 | per cent. per annum | $33\frac{1}{3}$ | 23·4498 |
| $3\frac{1}{2}$ | intereft, 1 l, or any | $28\frac{4}{7}$ | 20·1488 |
| 4 | other fum, will | 25 | 17·6730 |
| $4\frac{1}{2}$ | double itfelf in the | $22\frac{2}{9}$ | 15·7473 |
| 5 | following years. | 20 | 14·2067 |
| 6 | | $16\frac{2}{3}$ | 11·8957 |
| 7 | | $14\frac{2}{7}$ | 10·2448 |
| 8 | | $12\frac{1}{2}$ | 9·0065 |
| 9 | | $11\frac{1}{9}$ | 8·0432 |
| 10 | | 10 | 7·2725 |

The following Table will very much facilitate the calculation of the compound intereft of any fum, for any number of years, at various rates of intereft.

| Yrs. | 3 | $3\frac{1}{2}$ | 4 | $4\frac{1}{2}$ | 5 | 6 |
|---|---|---|---|---|---|---|
| | | | The Amounts of 1l. in any Number of Years. | | | |
| 1 | 1·0300 | 1·0350 | 1·0400 | 1·0450 | 1·0500 | 1·0600 |
| 2 | 1·0609 | 1·0712 | 1·0816 | 1·0920 | 1·1025 | 1·1236 |
| 3 | 1·0927 | 1·1087 | 1·1249 | 1·1412 | 1·1576 | 1·1910 |
| 4 | 1·1255 | 1·1475 | 1·1699 | 1·1925 | 1·2155 | 1·2625 |
| 5 | 1·1593 | 1·1877 | 1·2167 | 1·2462 | 1·2763 | 1·3382 |
| 6 | 1·1941 | 1·2293 | 1·2653 | 1·3023 | 1·3401 | 1·4185 |
| 7 | 1·2299 | 1·2723 | 1·3159 | 1·3609 | 1·4071 | 1·5036 |
| 8 | 1·2668 | 1·3168 | 1·3686 | 1·4221 | 1·4775 | 1·5939 |
| 9 | 1·3048 | 1·3629 | 1·4233 | 1·4861 | 1·5513 | 1·6895 |
| 10 | 1·3439 | 1·4106 | 1·4802 | 1·5530 | 1·6289 | 1·7909 |
| 11 | 1·3842 | 1·4600 | 1·5395 | 1·6229 | 1·7103 | 1·8983 |
| 12 | 1·4258 | 1·5111 | 1·6010 | 1·6959 | 1·7959 | 2·0122 |
| 13 | 1·4685 | 1·5640 | 1·6651 | 1·7722 | 1·8856 | 2·1329 |
| 14 | 1·5126 | 1·6187 | 1·7317 | 1·8519 | 1·9799 | 2·2609 |
| 15 | 1·5580 | 1·6753 | 1·8009 | 1·9353 | 2·0789 | 2·3966 |
| 16 | 1·6047 | 1·7340 | 1·8730 | 2·0224 | 2·1829 | 2·5404 |
| 17 | 1·6528 | 1·7947 | 1·9479 | 2·1134 | 2·2920 | 2·6928 |
| 18 | 1·7024 | 1·8575 | 2·0258 | 2·2085 | 2·4066 | 2.8543 |
| 19 | 1·7535 | 1·9225 | 2·1068 | 2·3079 | 2·5270 | 3·0256 |
| 20 | 1·8061 | 1·9898 | 2·1911 | 2·4117 | 2·6533 | 3·2071 |

The ufe of this Table, which contains all the powers $R^t$, to the 20th power, or the amounts of 1l, is chiefly to calculate the intereft, or the amount of any principal fum, for any time, not more than 20 years.

For example, let it be required to find, to how much 523l. will amount in 15 years, at the rate of 5 per cent. per annum compound intereft.

In the table, on the line 15, and in the column 5 per cent.

is the amount of 1l, viz. - - 2·0789
this multiplied by the principal 523

gives the amount - 1087·2647
or - - 1087l. 5s. 3¼d,
and therefore the intereft is 564l. 5s. 3¼d.

*Note* I. When the rate of intereft is to be determined to any other time than a year; as fuppofe to ½ a year, or ¼ a year, &c; the rules are ftill the fame; but then *t* will exprefs that time, and R muft be taken the amount for that time.

*Note* 2. When the compound intereſt, or amount, of any
ſum, is required for the parts of a year; it may be deter-
mined in the following manner :

1ſt, For any time which is ſome aliquot part of a year :—
Find the amount of 1l. for 1 year, as before; then that root
of it which is denoted by the aliquot part, will be the
amount of 1l. This amount being multiplied by the prin-
cipal ſum, will produce the amount of the given ſum as re-
quired.

2d, When the time is not an aliquot part of a year :—
Reduce the time into days, and take the 365th root of the
amount of 1l. for 1 year, which will give the amount of the
ſame for 1 day. Then raiſe this amount to that power
whoſe index is equal to the number of days, and it will be
the amount for that time. Which amount being multiplied
by the principal ſum, will produce the amount of that ſum
as before.

And in theſe calculations, the operation by logarithms will
be very uſeful.

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# OF ANNUITIES.

ANNUITY is a term uſed for any periodical income, ariſ-
ing from money lent, or from houſes, lands, ſalaries, pen-
ſions, &c, payable from time to time, but moſtly by annual
payments.

Annuities are divided into thoſe that are in Poſſeſſion, and
thoſe in Reverſion : the former meaning ſuch as have com-
menced ; and the latter ſuch as will not begin till ſome par-
ticular event has happened, or till after ſome certain time
has elapſed.

When an annuity is forborn for ſome years, or the pay-
ments not made for that time, the annuity is ſaid to be in
Arrears.

An annuity may alſo be for a certain number of years ; or
it may be without any limit, and then it is called a Perpetuity.

The Amount of an annuity, forborn for any number of
years, is the ſum ariſing from the addition of all the annui-
ties for that number of years, together with the intereſt due
upon each after it becomes due.

The

The Prefent Worth or Value of an annuity, is the price or fum which ought to be given for it, fuppofing it to be bought off, or paid all at once.

Let $a$ = the annuity, penfion, or yearly rent.
  $n$ = the number of years forborn, or lent for.
  $R$ = the amount of 1l. for 1 year.
  $m$ = the amount of the annuity.
  $v$ = its value, or its prefent worth.

Now, 1 being the prefent value of the fum $R$, by propor-tion the prefent value of any other fum $a$, is thus found:

as $R : 1 :: a : \dfrac{a}{R}$ the prefent value of $a$ due 1 year hence.

In like manner $\dfrac{a}{R^2}$ is the prefent value of $a$ due 2 years hence; for $R : 1 :: \dfrac{a}{R} : \dfrac{a}{R^2}$. So alfo $\dfrac{a}{R^3}, \dfrac{a}{R^4}, \dfrac{a}{R^5}$, &c, will be the prefent values of $a$, due at the end of 3, 4, 5, &c, years refpectively. Confequently the fum of all thefe, or

$\dfrac{a}{R} + \dfrac{a}{R^2} + \dfrac{a}{R^3} + \dfrac{a}{R^4} +$ &c, continued to $n$ terms, will be the prefent value of all the $n$ years annuities. And the va-lue of the perpetuity, is the fum of the feries to infinity.

But this feries, it is evident, is a geometrical progreffion, having $\dfrac{a}{R}$ both for its firft term and common ratio, and the number of its terms $n$; therefore the fum $v$ of all the terms, or the prefent value of all the annual payments, will be

$$v = \frac{a}{R-1} - \frac{a}{R-1} \times \frac{1}{R^n} \text{ or } = \frac{R^n - 1}{R - 1} \times \frac{a}{R^n}.$$

When the annuity is a perpetuity, $n$ being infinite; $R^n$ is alfo infinite, and therefore the laft quantity $\dfrac{1}{R^n}$ becomes $= 0$, therefore $\dfrac{a}{R-1} \times \dfrac{1}{R^n}$ alfo $= 0$; confequently the expref-fion becomes barely $v = \dfrac{a}{R-1}$; that is, any annuity divided by the intereft of 1l. for 1 year, gives the value of the per-petuity. So, if the rate of intereft be 5 per cent;

 Then $100a \div 5 = 20a$ is the value of the perpetuity at
  5 per cent:

Alfo $100a \div 4 = 25a$ is the value of the perpetuity at
  4 per cent:

And $100a \div 3 = 33\frac{1}{3}a$ is the value of the perpetuity at
  3 per cent: and fo on.

Again, becaufe the amount of 1l. in $n$ years, is $R^n$, its
increafe in that time will be $R^n - 1$; but its intereft for one
fingle year, or the annuity anfwering to that increafe, is
$R - 1$; therefore as $R - 1$ is to $R^n - 1$, fo is $a$ to $m$: that
is $m = \dfrac{R^n - 1}{R - 1} \times a$.

Hence, the feveral cafes relating to Annuities in Arrear,
will be refolved by the following equations:

$$m = \frac{R^n - 1}{R - 1} \times a = vR^n$$

$$v = \frac{R^n - 1}{R - 1} \times \frac{a}{R^n} = \frac{m}{R^n}$$

$$a = \frac{R - 1}{R^n - 1} \times m = \frac{R - 1}{R^n - 1} \times vR^n$$

$$n = \frac{\log. m - \log. v}{\log. R}$$

$$\text{Log. } R = \frac{\log. m - \log. v}{n}$$

$$r = \left(\frac{1}{R^n} - \frac{1}{R^p}\right) \times \frac{a}{R - 1}.$$

In this laft theorem, $r$ denotes the prefent value of an
annuity in reverfion, after $p$ years, or not commencing till
after the firft $p$ years, being found by taking the difference
between the two values $\dfrac{R^n - 1}{R - 1} \times \dfrac{a}{R^n}$ and $\dfrac{R^p - 1}{R - 1} \times \dfrac{a}{R^p}$, for
$n$ years and $p$ years.

## END of ALGEBRA,

### OR OF

## PART II. VOL. I.

# GEOMETRY:

BEING

PART III. of VOL. I.

# CONTENTS OF PART III.

## VOL. I.

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## GEOMETRY.

# GEOMETRY.

## DEFINITIONS.

1. A POINT is that which has pofition, but no magnitude, nor dimenfions ; neither length, breadth, nor thicknefs.

2. A Line is length, without breadth or thicknefs.

3. A Surface or Superficies, is an exten-fion, or a figure, of two dimenfions, length and breadth ; but without thicknefs.

4. A Body or Solid, is a figure of three dimenfions, namely, length, breadth, and thicknefs.
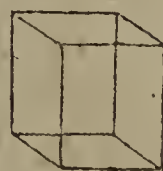
5. Lines are either Right, or Curved, or Mixed of thefe two.

6. A Right Line, or Straight Line, lies all in the fame direction, between its extremities ; and is the fhorteft diftance between two points.
When a Line is mentioned fimply, it means a Right Line.

7. A Curve continually changes its direc-tion between its extreme points.
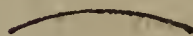
8. Lines are either Parallel, Oblique, Per-pendicular, or Tangential.

9. Parallel Lines are always at the fame perpendicular diftance ; and they never meet, though ever fo far produced.

10. Oblique right lines change their dif-tance, and would meet, if produced, on the fide of the leaft diftance.

11. One line is Perpendicular to another, when it inclines not more on the one fide

than

than the other; or when the angles on both fides of it are equal.

12. A line or circle is Tangential, or a Tangent to a circle, or other curve, when it touches it, without cutting, when both are produced.

13. An Angle is the inclination, or opening of two lines, having different directions, and meeting in a point.

14. Angles are Right or Oblique, Acute or Obtufe.

15. A Right Angle, is that which is made by one line perpendicular to another. Or when the angles on each fide are equal to one another, they are right angles.

16. An Oblique Angle, is that which is made by two oblique lines; and is either lefs or greater than a right angle.

17. An Acute Angle is lefs than a right angle.

18. An Obtufe Angle is greater than a right angle.

19. Superficies are either Plane or Curved.

20. A Plane Superficies, or a Plane, is that with which a right line may, every way, coincide. Or, if the line touch the plane in two points, it will touch it in every point. But if not, it is curved.

21. Plane figures are bounded either by right lines or curves.

22. Plane Figures that are bounded by right lines, have names according to the number of their fides, or of their angles; for they have as many fides as angles; the leaft number being three.

23. A figure of Three fides and angles, is called a Triangle. And it receives particular denominations from the relations of its fides and angles.

24. An Equilateral Triangle, is that whofe three fides are all equal.

25. An Ifofceles Triangle, is that which has two fides equal.

26. A

26. A Scalene Triangle, is that whofe three fides are all unequal.

27. A Right-angled Triangle, is that which has one right angle.

28. Other triangles are Oblique-angled, and are either Obtufe or Acute.

29. An Obtufe-angled Triangle, has one obtufe angle.

30. An Acute-angled Triangle, has all its three angles acute.

31. A figure of Four fides and angles, is called a Quadrangle, or a Quadrilateral.

32. A Parallelogram, is a quadrilateral which has both its pairs of oppofite fides parallel. And it takes the following particular names, viz. Rectangle, Square, Rhombus, Rhomboid.

33. A Rectangle is a parallelogram, having all its angles right ones.
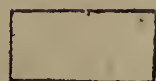
34. A Square is an equilateral rectangle; having all its fides equal, and its angles right ones.

35. A Rhomboid is an oblique-angled parallelogram.

36. A Rhombus is an equilateral rhomboid; having all its fides equal, but its angles oblique.

37. A Trapezium is a quadrilateral which hath not its oppofite fides parallel.

38. A Trapezoid hath only one pair of oppofite fides parallel.

39. A Diagonal is a right line joining any two oppofite angles of a quadrilateral.

40. Plane figures that have More than four fides are, in general, called Polygons: and they receive other particular names, according to the number of their fides or angles.

41. A Pentagon is a polygon of five fides; a Hexagon hath fix fides; a Heptagon, feven; an Octagon, eight; a Nonagon, nine; a Decagon, ten; an Undecagon, eleven; and a Dodecagon hath twelve fides.

42. A

42. A Regular Polygon hath all its fides and all its angles equal.—If they are not both equal, the polygon is Irregular.

43. An Equilateral Triangle is alfo a Regular Figure of three fides, and the Square is one of four; the former being alfo called a Trigon, and the latter a Tetragon.

44. A Circle is a plane figure bounded by a curve line, called the Circumference, which is every where equidiftant from a certain point within, called its Centre.

The circumference itfelf is often called a circle, and alfo the Periphery.

45. The Radius of a circle, is a right line drawn from the centre to the circumference.

46. The Diameter of a circle, is a right line drawn through the centre, and terminating in the circumference on both fides.

47. An Arc of a circle, is any part of the circumference.

48. A Chord, is a right line joining the extremities of an arc.

49. A Segment, is any part of a circle, bounded by an arc and its chord.

50. A Semicircle, is half the circle, or a fegment cut off by a diameter.

The half circumference is fometimes called the Semicircle.

51. A Sector, is any part of a circle which is bounded by an arc, and two radii drawn to its extremities;

52. A Quadrant, or Quarter of a circle, is a fector having a quarter of the circumference for its arc, and its two radii are perpendicular to each other.

A quarter of the circumference is fometimes called a Quadrant.
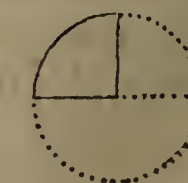
53. The

53. The Height or Altitude of a figure, is a perpendicular let fall from an angle, or its vertex, to the oppofite fide, called the bafe.
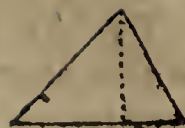
54. In a right angled triangle, the fide oppofite the right angle, is called the Hypothenufe ; and the other two fides are called the Legs, or fometimes the Bafe and Perpendicular.

55. When an angle is denoted by three letters, of which one ftands at the angular point, and the other two on the two fides, that which ftands at the angular point is read in the middle.
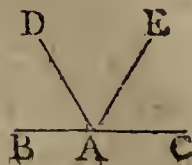
56. The circumference of every circle is fuppofed to be divided into 360 equal parts, called Degrees ; and each degree into 60 Minutes, each minute into 60 Seconds, and fo on. Hence a femicircle contains 180 degrees, and a quadrant 90 degrees.

57. The Meafure of a right-lined angle, is an arc of any circle contained between the two lines which form that angle, the angular point being the centre ; and it is eftimated by the number of degrees contained in that arc.

58. Right lines, or chords, are faid to be Equidiftant from the centre of a circle, when perpendiculars drawn to them from the centre are equal.

59. And the right line on which the Greater Perpendicular falls, is faid to be farther from the centre.

60. An Angle In a fegment, is that which is contained by two lines, drawn from any point in the arc of the fegment, to the two extremities of that arc.

61. An angle On a Segment, or an arc, is that which is contained by two lines, drawn from any point in the oppofite or fupplemental part of the circumference, to the extremities of the arc, and containing the arc between them.

62. An

62. An angle at the Circumference, is that whofe angular point is any where in the circumference. And an angle at the Centre, is that whofe angular point is at the centre.

63. A right-lined figure is Infcribed in a circle, or the circle Circumfcribes it, when all the angular points of the figure are in the circumference of the circle.

64. A right-lined figure Circumfcribes a circle, or the circle is Infcribed in it, when all the fides of the figure touch the circumference of the circle.

65. One right-lined figure is Infcribed in another, or the latter Circumfcribes the former, when all the angular points of the former are placed in the fides of the latter.

66. A Secant, is a line that cuts a circle, lying partly within, and partly without it.
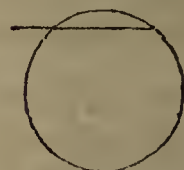
67. Identical figures, are fuch as have all the fides and all the angles of the one, refpectively equal to all the fides and all the angles of the other, each to each ; fo that if the one figure were applied to, or laid upon the other, all the fides of the one would exactly fall upon and cover all the fides of the other ; the two becoming as it were but one and the fame figure.

68. Similar figures, are thofe that have all the angles of the one equal to all the angles of the other, each to each, and the fides about the equal angles proportional.

69. The Perimeter of a figure, is the fum of all its fides taken together.

70. A Propofition, is fomething which is either propofed to be done, or to be demonftrated, and is either a problem or a theorem.

71. A Problem, is fomething propofed to be done.

72. A Theorem, is fomething propofed to be demonftrated.

73. A Lemma, is fomething which is premifed, or demonftrated, in order to render what follows more eafy.

74. A Corollary, is a confequent truth, gained immediately from fome preceding truth, or demonftration.

75. A Scholium, is a remark or obfervation made upon fomething going before it.

AXIOMS.

# AXIOMS.

1. THINGS which are equal to the fame thing, are equal to each other.

2. When equals are added to equals, the wholes are equal.

3. When equals are taken from equals, the remains are equal.

4. When equals are added to unequals, the wholes are unequal.

5. When equals are taken from unequals, the remains are unequal.

6. Things which are double of the fame thing, or equal things, are equal to each other.

7. Things which are halves of the fame thing, are equal.

8. The whole is equal to all its parts taken together.

9. Things which coincide, or fill the fame fpace, are identical, or mutually equal in all their parts.

10. All right angles are equal to one another.

11. Angles that have equal meafures, or arcs, are equal.

## THEOREM I.

IF two Triangles have Two Sides and the Included Angle in the one, equal to Two Sides and the Included Angle in the other, the Triangles will be Identical, or equal in all refpects.

In the two triangles ABC, DEF, if the fide AC be equal to the fide DF, and the fide BC equal to the fide EF, and the angle C equal to the angle F ; then will the two triangles be identical, or equal in all refpects.
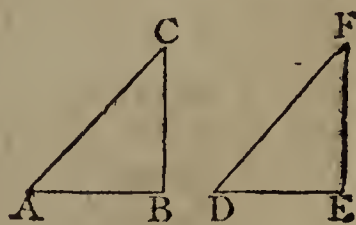
For conceive the triangle ABC to be applied to, or placed on, the triangle DEF, in fuch manner that the point C

may

may coincide with the point F, and the fide CA with the fide FD, which is equal to it.

Then, fince the angle F is equal to the angle C (by hyp.), the fide BC will fall on the fide EF. Alfo, becaufe AC is equal to DF, and BC equal to EF (by hyp.), the point A will coincide with the point D, and the point B with the point E ; confequently the fide AB will coincide with the fide DE ; therefore the two triangles are identical, and have all their other correfponding parts equal (ax. 9), namely, the fide AB equal to the fide DE, the angle A to the angle D, and the angle B to the angle E. Q. E. D.

## THEOREM II.

Triangles, which have Two Angles, and the Side which lies between them, equal, are Identical, or have their other fides and Angle equal.

Let the two triangles ABC, DEF, have the angle A equal to the angle D, the angle B equal to the angle E, and the fide AB equal to the fide DE ; then thefe two triangles will be identical.
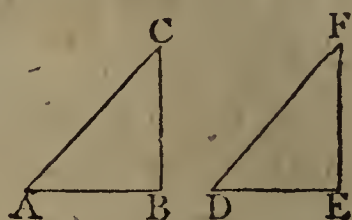
For, conceive the triangle ABC to be placed on the triangle DEF, in fuch manner that the fide AB may fall exactly on the equal fide DE. Then, fince the angle A is equal to the angle D (by hyp.), the fide AC muft fall on the fide DF : and, in like manner, becaufe the angle B is equal to the angle E, the fide BC muft fall on the fide EF. Thus the three fides of the triangle ABC will be exactly placed on the three fides of the triangle DEF ; and confequently the two triangles are identical (ax. 9), having the other two fides AC, BC equal to the two DF, EF, and the remaining angle C equal to the remaining angle F. Q. E. D.

## THEOREM III.

In an Ifofceles Triangle, the Angles at the Bafe are equal. Or, if a Triangle have Two Sides equal, their Oppofite Angles will alfo be equal.

If the triangle ABC have the fide AC equal to the fide BC : then will the angle B be equal to the angle A.

For, conceive the angle C to be bifected, or divided into two equal parts, by the line CD, making the angle ACD equal to the angle BCD.
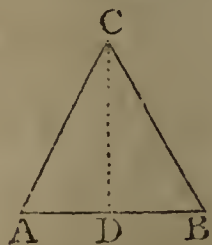
Then,

Then, the two triangles ACD, BCD, have two fides and the contained angle of the one, equal to two fides and the contained angle of the other, viz. the fide AC equal to BC, the angle ACD equal to BCD, and the fide CD common ; therefore thefe two triangles are identical, or equal in all refpects (th. 1.); and confequently the angle A equal to the angle B.   Q. E. D.

*Corol.* 1. Hence, the line which bifects the vertical angle of an ifofceles triangle, bifects the bafe, and is alfo perpendicular to it.

*Corol.* 2. Hence too it appears, that every equilateral triangle is alfo equiangular, or has all its angles equal.

## THEOREM IV.

If a Triangle have Two of its Angles equal to each other, the Sides Oppofite to them will alfo be equal.

IF the triangle ABC, have the angle A equal to the angle B, it will alfo have the fide AC equal to the fide BC.

For, conceive the fide AB to be bifected in the point D, making AD equal to DB ; and join DC, dividing the whole triangle into the two triangles ACD, BCD. Alfo, conceive the triangle ACD to be turned over upon the triangle BCD, fo that AD may fall on BD.

Then, becaufe the line AD is equal to the line DB (by hyp.), the point A coincides with the point B, and the point D with the point D. Alfo, becaufe the angle A is equal to the angle B (by hyp.), the line AC will fall on the line BC, and the extremity C of the fide AC will coincide with the extremity C of the fide BC, becaufe DC is common to both ; confequently the fide AC is equal to BC.   Q. E. D.

*Corol.* Hence every equiangular triangle, is alfo equilateral.

## THEOREM V.

TRIANGLES which have their Three Sides mutually equal, are Identical, or have alfo their Three Angles equal, each to each.

Let the two triangles ABC, ABD, have their three fides refpectively equal, viz. the fide AB equal to AB, AC to AD, and BC to BD ; then fhall the two triangles be identical, or have their angles equal, viz.
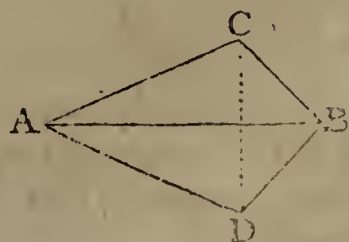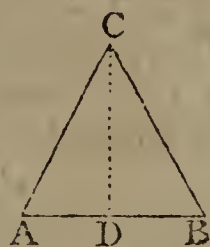
thofe angles that are oppofite to the equal fides ; namely, the angle BAC to the angle BAD, the angle ABC to the angle ABD, and the angle C to the angle D.

For, conceive the two triangles to be joined together by their longeft equal fides ; and draw the line CD.

Then, in the triangle ACD, becaufe the fide AC is equal to AD (by hyp.), the angle ACD is equal to the angle ADC (th. 2). In like manner, in the triangle BCD, the angle BCD is equal to the angle BDC, becaufe the fide BC is equal to BD. Hence then, the angle ACD being equal to the angle ADC, and the angle BCD to the angle BDC, by equal additions, the fum of the two angles ACD, BCD, is equal to the fum of the two ADC, BDC, (ax. 2), that is, the whole angle ACB equal to the whole angle ADB.

Since then, the two fides AC, CB, are equal to the two fides AD, DB, each to each (by hyp.) and their contained angles ACB, ADB, alfo equal, the two triangles ABC, ABD, are identical (th. 1), and have the other angles equal, viz. the angle BAC to the angle BAD, and the angle ABC to the angle ABD.    Q. E. D.

## THEOREM VI.

THE Angles which one Right Line makes with another, on the Same Side of it, are together equal to Two Right Angles.

Let the line AB meet the line CD ; then will the two angles ABC, ABD, taken together, be equal to two right angles.

For, firft, when the two angles ABC, ABD, are equal to each other, they are both of them right angles (def. 15).

But when the angles are unequal, fuppofe BE drawn perpendicular to CD. Then, fince the two angles EBC, EBD, are right angles (def. 15.) and the angle EBD is equal to the two angles EBA, ABD together (ax. 8), the three angles, EBC, EBA, and ABD, are equal to two right angles.

But the two angles EBC, EBA, are together equal to the angle ABC (ax. 8). Confequently the two angles ABC, ABD, are alfo equal to two right angles.    Q. E. D.

Corol. 1. Hence alfo, converfely, if the two angles ABC, ABD, on both fides of the line AB, make up together two right angles, then CB and BD form one continued right line CD.
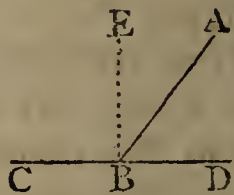
*Corol.* 2. Hence, all the angles which can be made, at any point B, by any number of lines, on the same side of the right line CD, are, when taken all together, equal to two right angles.

*Corol.* 3. And, as all the angles that can be made on the other side of the line CD are also equal to two right angles ; therefore all the angles that can be made quite round a point B, by any number of lines, are equal to four right angles.

*Corol.* 4. Hence also the whole circumference of a circle, being the sum of the measures of all the angles that can be made about the centre F (def. 57), is the measure of four right angles. Consequently a semicircle, or 180 degrees, is the measure of two right angles ; and a quadrant, or 90 degrees, the measure of one right angle.



## THEOREM VII.

If two Lines Interfect each other, the Oppofite Angles will be equal.

LET the two lines AB, CD interfect in the point E ; then will the angle AEC be equal to the angle BED, and the angle AED equal to the angle CEB.
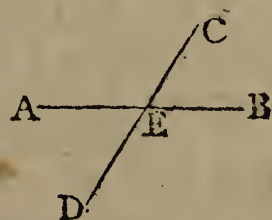


For, fince the line CE meets the line AB, the two angles AEC, BEC, taken together, are equal to two right angles (th. 6).

In like manner, the line BE, meeting the line CD, makes the two angles CEB, DEB equal to two right angles.

Therefore the fum of the two angles AEC, BEC, is equal to the fum of the two CEB, DEB (ax. 1).

And if the angle CEB, which is common, be taken away from them both, the remaining angle AEC will be equal to the remaining angle BED (ax. 3).

And in like manner it may be fhewn, that the angle AED is equal to the oppofite angle BEC.

## THEOREM VIII.

IF One Side of a Triangle be produced, the Outward Angle will be Greater than either of the two Inward Oppofite Angles.

Let

Let ABC be a triangle, having the
fide AB produced to D ; then will the
outward angle CBD be greater than
either of the inward oppofite angles A
or C.

For, conceive the fide BC to be bi-
fected in the point E, and draw the line
AE, producing it till EF be equal to
AE : and join BF.
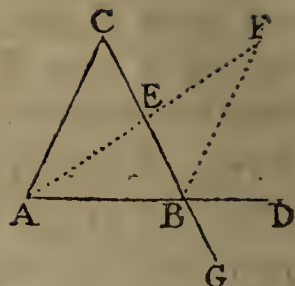
Then, fince the two triangles AEC, BEF have the fide
AE = the fide EF, and the fide BE = the fide CE (by
fuppof.) and the included or oppofite angles at E alfo equal.
(th. 7.), therefore thofe two triangles are equal in all re-
fpects (th. 1.), and have the angle C = the correfponding
angle EBF. But the angle CBD is greater than the angle
EBF ; confequently the faid outward angle CBD is alfo
greater than the angle C.

In like manner, if CB be produced to G, and AB be
bifected, it may be fhewn that the outward angle ABG, or
its equal CBD, is greater than the other angle A.

### THEOREM IX.

THE Greater Side, of every Triangle, is Oppofite to
the Greater Angle ; and the Greater Angle oppofite to the
Greater Side.

Let ABC be a triangle, having the fide
AB greater than the fide AC ; then will
the angle ACB, oppofite the greater fide
AB, be greater than the angle B, oppofite
the lefs fide AC.
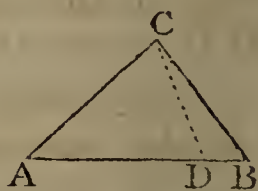
For, on the greater fide AB take the
part AD equal to the lefs fide AC, and join CD. Then,
fince BCD is a triangle, the outward angle ADC is greater
than the inward oppofite angle B (th. 8). But the angle
ACD is equal to the faid outward angle ADC, becaufe AD
is equal to AC (th. 3). Confequently the angle ACD alfo
is greater than the angle B. And fince the angle ACD is
only a part of ACB, much more muft the whole angle
ACB be greater than the angle B. Q. E. D.

Again, converfely, if the angle C be greater than the
angle B, then will the fide AB, oppofite the former, be
greater than the fide AC oppofite the latter.

For, if AB be not greater than AC, it muft be either
equal to it, or lefs than it. But it cannot be equal, for
then

then the angle C would be equal to the angle B (th. 3.); which it is not, by the suppofition. Neither can it be lefs, for then the angle C would be lefs than the angle B, by the former part of this; which is alfo contrary to the fuppofition. The fide AB, then, being neither equal to AC, nor lefs than it, muft neceffarily be greater. Q. E. D.

## THEOREM X.

The Sum of any Two Sides of a Triangle, is Greater than the Third Side.

LET ABC be a triangle; then will the fum of any two of its fides be greater than the third fide, as for inftance, AC + CB greater than AB.
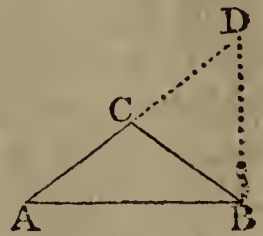
For, produce AC till CD be equal to CB, or AD equal to the fum of the two AC + CB; and join BD.—Then, becaufe CD is equal to CB (by conftr.), the angle D is equal to the angle CBD (th. 3). But the angle ABD is greater than the angle CBD, confequently it muft alfo be greater than the angle D. And, fince the greater fide of any triangle is oppofite to the greater angle (th. 9), the fide AD (of the triangle ABD) is greater than the fide AB. But AD is equal to AC and CD, or AC and CB, taken together (by conftr.); therefore AC + CB is alfo greater than AB. Q. E. D.

*Corol.* The fhorteft diftance between two points, is a fingle right line drawn from the one point to the other.

## THEOREM XI.

The Difference of any Two Sides of a Triangle is Lefs than the Third Side.

LET ABC be a triangle; then will the difference of any two fides, as AB — AC, be lefs than the third fide BC.
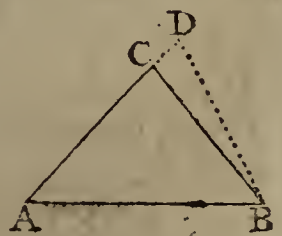
For, produce the lefs fide AC to D, till AD be equal to the greater fide AB, fo that CD may be the difference of the two fides AB — AC; and join BD.— Then, becaufe AD is equal to AB (by conftr.), the oppofite angles D and ABD are equal (th. 3). But the angle CBD is lefs than the angle ABD, and confequently alfo lefs than the equal angle D. And fince the greater fide of any triangle

is

is oppofite to the greater angle (th. 9), the fide CD (of the triangle BCD) is lefs than the fide BC.　Q. E. D.

<center>THEOREM XII.</center>

If a Line Interfect two Parallel Lines, it will make the Alternate Angles Equal to each other.

LET the line EF cut the two parallel lines AB, CD ; then will the angle AEF be equal to the alternate angle EFD.
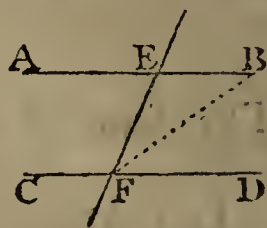
For if they are not equal, one of them muft be greater than the other ; let it be EFD for inftance which is the greater, if poffible; and conceive the line FB to be drawn cutting off the part or angle EFB equal to the angle AEF, and meeting the line AB in the point B.

Then, fince the outward angle AEF, of the triangle BEF, is greater than the inward oppofite angle EFB (th. 8) ; and fince thefe two angles alfo are equal (by the conftr.) it follows, that thofe angles are both equal and unequal at the fame time : which is impoffible.　Therefore the angle EFD is not unequal to the alternate angle AEF, that is, they are equal to each other.　Q. E. D.

Corol. Right lines which are perpendicular to one, of two parallel lines, are alfo perpendicular to the other.

<center>THEOREM XIII.</center>

IF a Line, Cutting Two other Lines, make the Alternate Angles Equal to each other, thofe two Lines will be Parallel.

Let the line EF, cutting the two lines AB, CD, make the alternate angles AEF, DFE equal to each other ; then will AB be parallel to CD.
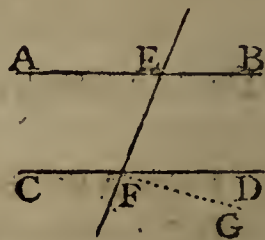
For if they be not parallel, let fome other line, as FG, be parallel to AB. Then, becaufe of thefe parallels, the angle AEF is equal to the alternate angle EFG (th. 12).　But the angle AEF is equal to the angle EFD (by hyp.)　Therefore the angle EFD is equal to the angle EFG (ax. 1,) that is, a part is equal to the whole, which is impoffible.　Therefore no line but CD can be parallel to AB.　Q. E. D.

Corol. Thofe lines which are perpendicular to the fame line, are parallel to each other.

<div align="right">THE-</div>

<center>I</center>

## THEOREM XIV.

IF a Line Cut two Parallel Lines; the Outward Angle will be Equal to the Inward Oppolite one, on the Same Side; and the two Inward Angles, on the Same Side, will be equal to Two Right Angles.

Let the line EF cut the two parallel lines AB, CD; then will the outward angle EGB be equal to the inward oppolite angle GHD, on the fame fide of the line EF; and the two inward angles BGH, GHD, taken together, will be equal to two right angles.
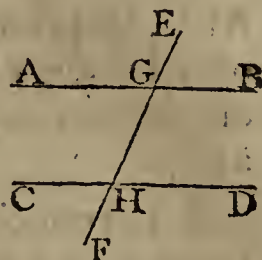
For, fince the two lines AB, CD are parallel, the angle AGH is equal to the alternate angle GHD (th. 12). But the angle AGH is equal to the oppolite angle EGB (th. 7). Therefore the angle EGB is alfo equal to the angle GHD (ax. 1). Q. E. D.

Again, becaufe the two adjacent angles EGB, BGH are together equal to two right angles (th. 6); of which the angle EGB has been fhewn to be equal to the angle GHD; therefore the two angles BGH, GHD, taken together, are alfo equal to two right angles.

*Corol.* 1. And, converfely, if one line meeting two other lines, make the angles on the fame fide of it equal, thofe two lines are parallels.

*Corol.* 2. If a line, cutting two other lines, make the fum of the two inward angles, on the fame fide, lefs than two right angles, thofe two lines will not be parallel, but will meet each other when produced.

## THEOREM XV.

Thofe Lines which are Parallel to the Same Line, are Parallel to each other.

LET the lines AB, CD be each of them parallel to the line EF; then fhall the lines AB, CD be parallel to each other.
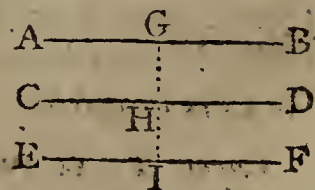
For, let the line GI be perpendicular to EF. Then will this line be alfo perpendicular to both the lines AB, CD (corol. th. 12), and confequently the two lines AB, CD are parallels (corol. th. 13).

Q. E. D.

## THEOREM XVI.

IF one Side of a Triangle be produced, the Outward Angle will be equal to both the Inward Oppofite Angles taken together.

Let the fide AB, of the triangle ABC, be produced to D; then will the outward angle CBD be equal to the fum of the two inward oppofite angles A and C.
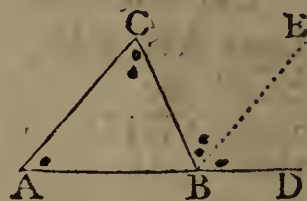
For, conceive BE to be drawn parallel to the fide AC of the triangle. Then BC, meeting the two parallels AC, BE, makes the alternate angles C and CBE equal (th. 12). And AD, cutting the fame two parallels AC, BE, makes the inward and outward angles on the fame fide, A and EBD, equal to each other (th. 14). Therefore, by equal additions, the fum of the two angles A and C, is equal to the fum of the two CBE and EBD, that is to the whole angle CBD (by ax. 2). Q. E. D.

## THEOREM XVII.

The Sum of all the Three Angles of any Plane Triangle, is equal to Two Right Angles.

LET ABC be any plane triangle; then the fum of the three angles A + B + C is equal to two right angles.
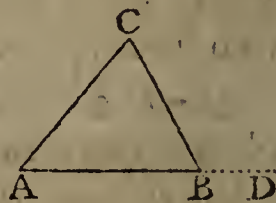
For, let the fide AB be produced to D. Then the outward angle CBD is equal to the fum of the two inward oppofite angles A + C (th. 16). To each of thefe equals add the inward angle B, then will the fum of the three inward angles A + B + C be equal to the fum of the two adjacent angles ABC + CBD (ax. 2). But the fum of thefe two laft adjacent angles is equal to two right angles (th. 6). Therefore alfo the fum of the three angles of the triangle A + B + C is equal to two right angles (ax. 1). Q. E. D.

*Corol.* 1. If two angles in one triangle, be equal to two angles in another triangle, the third angles will alfo be equal (ax. 3), and the two triangles equiangular.

*Corol.* 2. If one angle in one triangle, be equal to one angle in another, the fums of the remaining angles will alfo be equal (ax. 2).

*Corol.* 3.

*Corol.* 3. If one angle of a triangle be right, the fum of the other two will alfo be equal to a right angle, and each of them fingly will be acute, or lefs than a right angle.

*Corol.* 4. The two leaft angles of every triangle are acute, or each lefs than a right angle.

## THEOREM XVIII.

In any Quadrangle, the Sum of all the Four Inward Angles, is equal to Four Right Angles.

LET ABCD be a quadrangle; then the fum of the four inward angles, A + B + C + D is equal to 4 right angles.
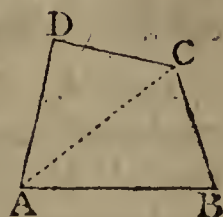
Let the diagonal AC be drawn, dividing the quadrangle into two triangles, ABC, ADC. Then, becaufe the fum of the three angles of each of thefe triangles, is equal to two right angles (th. 17); it follows, that the fum of all the angles of both triangles, which make up the four angles of the quadrangle, muft be equal to 4 right angles (ax. 2).

Q. E. D.

*Corol.* 1. Hence, if three of the angles be right ones, the fourth will alfo be a right angle.

*Corol.* 2. Alfo, if the fum of two of the four angles be equal to two right angles, the fum of the remaining two will likewife be equal to two right angles.

## THEOREM XIX.

IN any Polygon, the Sum of all the Inward Angles, taken together, is equal to Twice as many Right Angles, wanting four, as the Figure has Sides.

Let ABCDE be any polygon; the fum of all its inward angles, A + B + C + D + E, is equal to twice as many right angles, wanting four, as the figure has fides.
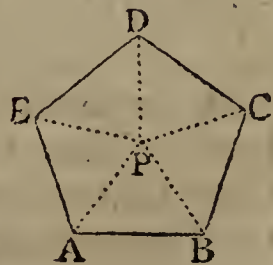
For, from any point P, within it, draw lines PA, PB, PC, &c, to all the angles, dividing the polygon into as many triangles as it has fides. Now the fum of the three angles of each of thefe triangles, is equal to two right angles (th. 17); therefore the fum of the angles of all the triangles is equal to twice as many right angles as the figure has fides. But the fum of all the angles about the point P, which are fo
many

many of the angles of the triangles, but no part of the inward angles of the polygon, is equal to four right angles (corol. 3, th. 6), and muſt be deducted out of the former ſum. Hence it follows that the ſum of all the inward angles of the polygon alone A + B + C + D + E, is equal to twice as many right angles as the figure has ſides, wanting the ſaid four right angles. Q. E. D.

### THEOREM XX.

IF every Side of any right-lined Figure be produced out, the Sum of all the Outward Angles thereby made, will be equal to Four Right Angles.

Let A, B, C, &c. be the outward angles of any polygon, made by producing all the ſides; then will the ſum A + B + C + D + E, of all thoſe outward angles, be equal to four right angles.
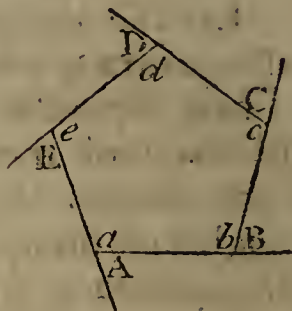
For, every one of theſe outward angles, together with its adjacent inward angle, make up two right angles, as A + a equal to two right angles, being the two angles made by one line meeting another (th. 6). And there being as many outward, or inward angles, as the figure has ſides. Therefore the ſum of all the inward and outward angles, is equal to twice as many right angles as the figure has ſides. But the ſum of all the inward angles, with four right angles, is equal to twice as many right angles as the figure has ſides (th. 19). Therefore the ſum of all the inward and all the outward angles, is equal to the ſum of all the inward angles and four right angles (by ax. 1). From each of theſe take away all the inward angles, and there remains all the outward angles equal to four right angles (by ax. 3).

### THEOREM XXI.

A Perpendicular is the Shorteſt Line that can be drawn from a Given Point to an Indefinite Line. And, of any other Lines drawn from the ſame Point, thoſe that are Neareſt the Perpendicular, are Leſs than thoſe More Remote.

IF AB, AC, AD, &c, be lines drawn from the given point A, to the indefinite line DE, of which AB is perpendicular. Then ſhall the perpendicular AB be leſs than AC, and AC leſs than AD, &c.
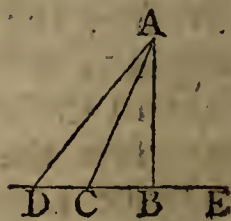
For, the angle B being a right one, the
angle

angle C is acute (by cor. 3, th. 17), and therefore lefs than the angle B. But the greater angle of a triangle is fub-tended by the greater fide (th. 9). Therefore the fide AC is greater than the fide AB.

Again, the angle ACB being acute, as before, the adjacent angle ACD will be obtufe (by th. 6). Bnt the angle D is acute (corol. 3, th. 17); Therefore the angle ACD is greater than the angle D. And, fince the greater fide is oppofite to the greater angle, therefore the fide AD is greater than the fide AC. Q. E. D.

*Corol.* A perpendicular is the leaft diftance of a given point from a line.

### THEOREM XXII.

THE Oppofite Sides and Angles of any Parallelogram are equal to each other ; and the Diagonal divides it into two Equal Parts.

Let ABCD be a parallelogram, of which the diagonal is BD ; then will its oppofite fides and angles be equal to each other, and the diagonal BD will divide it into two equal parts, or triangles.
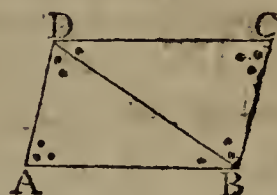
For, fince the fides AB and DC are parallel, as alfo the fides AD and BC (defin. 32), and the line BD meets them ; therefore the alternate angles are equal (th. 12), namely, the angle ABD to the angle CDB, and the angle ADB to the angle CBD. Hence the two triangles, having two angles in the one equal to two angles in the other, have alfo their third angles equal (cor. 1. th. 17), namely, the angle A equal to the angle C, which are two of the oppofite angles of the parallelogram.

Alfo, if to the equal angles ABD, CDB, be added the equal angles CBD, ADB, the wholes will be equal (ax. 2), namely, the whole angle ABC to the whole ADC, which are the other two oppofite angles of the parallelogram.

Q. E. D.

Again, fince the two triangles are mutually equiangular, and have a fide in each equal, viz. the common fide BD ; therefore the two triangles are identical (th 2), or equal in all refpeâs, namely, the fide AB to the oppofite fide DC, and AD to the oppofite fide BC, and the whole triangle ABD to the whole triangle BCD. Q. E. D.

*Corol.* 1.

*Corol.* 1. Hence, if one angle of a parallelogram be a right angle, all the other three will alſo be right angles, and the parallelogram a rectangle.

*Cor l.* 2. Hence alſo, the ſum of any two adjacent angles of a parallelogram is equal to two right angles.

### THEOREM XXIII.

Every Quadrilateral whoſe Oppoſite Sides are Equal, is a Parallelogram.

Let ABCD be a quadrangle, having the oppoſite ſides equal, namely, the ſide AB equal to DC, and AD equal to BC ; then ſhall theſe equal ſides be alſo parallel, and the figure a parallelogram.
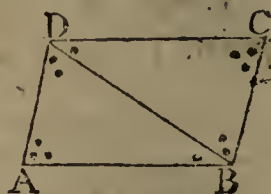
For, let the diagonal BD be drawn. Then, the triangles ABD, CBD being mutually equilateral (by hyp.), they are alſo mutually equiangular (th. 5) ; and conſequently the oppoſite ſides are parallel (th. 13) ; viz. the ſide AB parallel to DC, and AD parallel to BC, and the figure is a parallelogram. Q. E. D.

### THEOREM XXIV.

Right Lines joining the Correſpoding Extremes of two Equal and Parallel Lines, are themſelves Equal and Parallel.

Let AB, DC be two equal and parallel lines; then will the lines AD, BC, which join their extremes, be alſo equal and parallel. {See the fig. above.}

For, draw the diagonal BD. Then, becauſe AB and DC are parallel (by hyp.), the angle ABD is equal to the alternate angle BDC (th. 12). Hence then, the two triangles having two ſides and the contained angles equal, viz. the ſide AB equal to the ſide DC, and the ſide BD common, and the contained angle ABD equal to the contained angle BDC; they will have the remaining ſides and angles alſo reſpectively equal (th. 1) ; conſequently AD equal to BC, and alſo parallel to it (th. 12). Q. E. D.

### THEOREM XXV.

Parallelograms, or Triangles, ſtanding on the Same Baſe, and between the Same Parallels, are equal to each other.

Let

Let ABCD, ABEF be two parallelo-
grams, and ABC, ABF two triangles,
ftanding on the fame bafe AB, and be-
tween the fame parallels AB, DE ; then
will the parallelogram ABCD be equal to
the parallelogram ABEF, and the triangle
ABC to the triangle ABF.
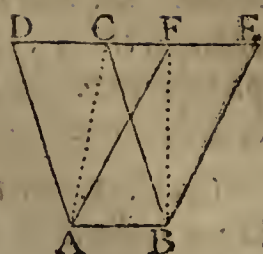
For, fince the line DE cuts the two
parallels AF, BE, and the two AD, BC, it makes the angle
E equal to the angle AFD, and the angle D equal to the,
angle BCE (th. 14) ; the two triangles ADF, BCE are
therefore equiangular (cor. 1, th. 17) ; and having the two
correfponding fides AD, BC equal (th. 22), being oppofite
fides of a parallelogram, thefe two triangles are identical,
or equal in all refpects (th. 2). If each of thefe equal tri-
angles then be taken from the whole fpace ABED, there will
remain the parallelogram ABEF in the one cafe, equal to
the parallelogram ABCD in the other (by ax. 3).

Alfo, the triangles ABC, ABF, on the fame bafe AB, and
between the fame parallels, are equal, being the halves of
the faid equal parallelograms (th. 22). Q. E. D.

*Corol.* 1. Parallelograms, or triangles, having the fame
bafe and altitude, are equal. For the altitude is the fame
as the perpendicular or diftance between the two parallels,
which is every where equal, by the definition of parallels.

*Corol.* 2. Parallelograms, or triangles, having equal bafes
and altitudes, are equal. For, if the one figure be applied
with its bafe upon the other, the bafes will coincide or be
the fame, becaufe they are equal; and fo the two figures,
having the fame bafe and altitude, are equal.

## THEOREM XXVI.

IF a Parallelogram and a Triangle ftand on the Same
Bafe, and between the Same Parallels, the Parallelogram
will be Double the Triangle, or the Triangle Half the Pa-
rallelogram.

Let ABCD be a parallelogram, and ABE
a triangle, on the fame bafe AB, and between
the fame parallels AB, DE ; then will the pa-
rallelogram ABCD be double the triangle
ABE, or the triangle half the parallelogram.
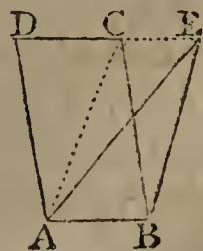
For, draw the diagonal AC of the paral-
lelogram, dividing it into two equal parts
(th. 22). Then, becaufe the triangles ABC,
ABE,

ABE, on the fame bafe, and between the fame parallels, are equal (th. 25) ; and becaufe the triangle ABC is half the parallelogram ABCD (th. 22), the other equal triangle ABE is alfo equal to half the fame parallelogram ABCD.   Q. E. D.

*Corol.* 1.  A triangle is equal to half a parallelogram of the fame bafe and altitude, becaufe the altitude is the perpendicular diftance between the parallels, which is every where equal, by the definition of parallels.

*Corol.* 2.  If the bafe of a parallelogram be half that of a triangle, of the fame altitude, or the bafe of the triangle be double that of the parallelogram, the two figures will be equal to each other.

## THEOREM XXVII.

RECTANGLES that are contained under Equal Lines, are Equal to each other.

Let BD, FH be two rectangles, having the fides AB, BC, equal to the fides EF, FG, each to each ; then will the rectangle BD be equal to the rectangle FH.
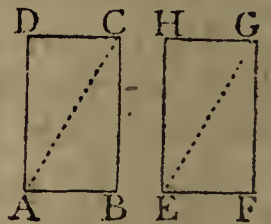
For, draw the two diagonals AC, EG, dividing the two parallelograms each into two equal parts.   Now, the two triangles ABC, EFG, are equal to each other (th. 1), becaufe they have the two fides AB, BC, and the contained angle B equal to the two fides EF, FG, and the contained angle F (by hyp.)   But thefe equal triangles are the halves of the refpective rectangles.   And becaufe the halves, or the triangles, are equal, the wholes, or the rectangles DB, HF, are equal alfo (by ax. 6).   Q. E. D.

*Corol.*  The fquares on equal lines, are alfo equal ; for every fquare is a fpecies of rectangle.

## THEOREM XXVIII.

THE Complements of the Parallelograms, which are about the Diagonal of any Parallelogram, are equal to each other.

Let AC be a parallelogram, BD a diagonal, EIF parallel to AB or DC, and GIH parallel to AD or BC, making AI, IC complements to the parallelograms EG, HF, which are about the diagonal DB: then will the complement AI be equal to the complement IC.

For,
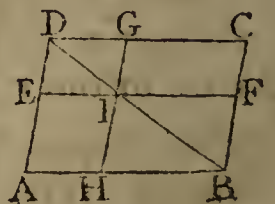
For, fince the diagonal DB bifects the three parallelograms AC, EG, HF (th. 22) ; therefore, the whole triangle DAB. being equal to the whole triangle DCB, and the parts DEI, IHB refpectively equal to the parts DGI, IFB, the remaining parts AI, IC muft alfo be equal (by ax. 3). Q. E. D.

### THEOREM XXIX.

A Trapezoid, or Trapezium having two Sides Parallel, is equal to Half a Parallelogram, whofe Bafe is the Sum of thofe two Sides, and its Altitude the Perpendicular Diftance between them,

Let ABC D be the trapezoid, having its two fides AB, DC parallel ; and in AB produced take BE equal to DC, fo that AE may be the fum of the two parallel fides ; produce DC alfo, and let EF, GC BH be all three parallel to AD. Then is AF a parallelogram of the fame altitude with the trapezoid ABCD, having its bafe AE equal to the fum of the parallel fides of the trapezoid ; and it is to be proved that the trapezoid ABCD is equal to half the parallelogram AF.
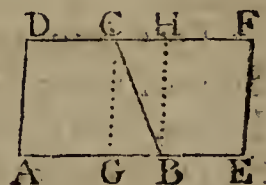
Now, fince triangles, or parallelograms, of equal bafes and altitude, are equal (corol. 2. th. 25), the parallelogram DG is equal to the parallelogram HE, and the triangle CGB equal to the triangle CBH ; confequently the line BC bifects, or equally divides, the parallelogram AF, and ABCD is the half of it. Q. E. D.

### THEOREM XXX.

The Sum of all the Rectangles contained under one Whole Line, and the feveral Parts of another Line, any how divided, is Equal to the Rectangle contained under the Two Whole Lines.

Let AD be the one line, and AB the other line, divided into the parts AE, EF, FB ; then fhall the rectangle contained by AD and AB, be equal to the fum of the rectangles of AD and AE, AD and EF, and AD and FB ; thus exprefled AD . AB $= AD . AE + AD . EF + AD . FB.$
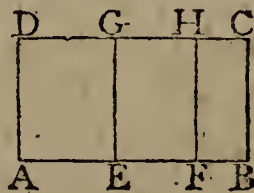
For, make the rectangle AC of the two whole lines AD, AB ; and draw EG, FH, perpendicular to AB, or parallel to AD, to which they are equal (th. 22). Then the whole rectangle AC is made up of all the other rectangles AG, EH,

EH, FC. But thefe rectangles are contained by AD and AE, EG and EF, FH and FB; which are equal to the rectangles of AD and AE, AD and EF, AD and FB, becaufe AD is equal to each of the two EG, FH. Therefore the rectangle AD . AB is equal to the fum of all the other rectangles AD . AE, AD . EF, AD . FB.

*Corol.* If a right line be divided into any two parts; the fquare on the whole line, is equal to both the rectangles of the whole line and each of the parts.

## THEOREM XXXI.

THE Square of the Sum of two Lines, is greater than the Sum of their Squares, by Twice the Rectangle of the faid Lines. Or, the Square of a whole Line, is equal to the Squares of its two Parts, together with Twice the Rectangle of thofe Parts.

Let the line AB be the fum of any two lines AC, CB; then will the fquare of AB be equal to the fquares of AC, CB, together with twice the rectangle of AC, CB. That is, $AB^2 = AC^2 + CB^2 + 2AC . CB$.
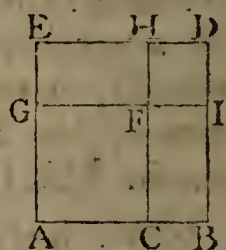
For, let ABDE be the fquare on the fum or whole line AB, and ACFG the fquare on the part AC. Produce CF and GF to the other fides at H and I.

From the lines CH, GI, which are equal, being each equal to the fides of the fquare AB or BD (th. 22), take the parts CF, GF, which are equal alfo, being the fides of the fquare AF, and there remains FH equal to FI, which are alfo equal to DH, DI, being the oppofite fides of a parallelogram. Hence the figure HI is equilateral; and it has all its angles right ones (corol. 1, th. 22); it is therefore a fquare on the line FI, or the fquare of its equal CB. Alfo the figures EF, FB are equal to two rectangles under AC and CB, becaufe GF is equal to AC, and FH or FI equal to CB. But the whole fquare AD is made up of the four figures, viz. the two fquares AF, FD, and the two equal rectangles EF, FB. That is, the fquare of AB is equal to the fquares of AC, CB, together with twice the rectangle of AC, CB.

*Corol.* Hence, if a line be divided into two equal parts; the fquare of the whole line, will be equal to four times the fquare of half the line.

## THEOREM XXXII.

THE Square of the Difference of two lines, is lefs than the Sum of their Squares, by Twice the Rectangle of the faid Lines.

Let AC, BC be any two lines, and AB their difference: then will the fquare of AB be lefs than the fquares of AC, BC, by twice the rectangle of AC and BC. Or, $AB^2 = AC^2 + BC^2 - 2AC. BC.$

For, let ABDE be the fquare on the difference AB, and ACFG the fquare on the line AC. Produce ED to H; alfo produce DB and HC, and draw KI, making BI the fquare of the other line BC.
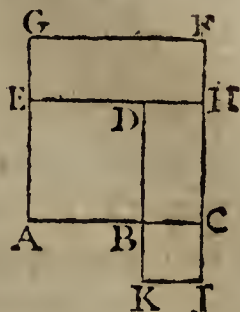
Now it is vifible that the fquare AD is lefs than the two fquares AF, BI, by the two rectangles EF, DI. But GF is equal to the one line AC, and GE or FH is equal to the other line BC; confequently the rectangle EF, contained under EG and GF, is equal to the rectangle of AC and BC.

Again FH being equal to CI or BC or DH, by adding the common part HC, the whole HI will be equal to the whole FC or equal to AC; and confequently the figure DI is equal to the rectangle contained by AC and BC.

Hence the two figures EF, DI are two rectangles of the two lines AC, BC; and confequently the fquare of AB is lefs than the fquares of AC, BC, by twice the rectangle AC . BC.

## THEOREM XXXIII.

THE Difference of the Squares of any two unequal Lines, is equal to the Rectangle under the Sum and Difference of the fame Lines.

Let AB, AC be any two unequal lines; then will the difference of the fquares of AB, AC be equal to a rectangle under their fum and difference. That is, $AB^2 - AC^2 = \overline{AB + AC} . \overline{AB - AC}.$

For, let ABDE be the fquare of AB, and ACFG the fquare of AC. Produce DB till BH be equal to AC; draw HI parallel to AB or ED, and produce FC both ways to I and K.

VOL. I.                    X                    Then
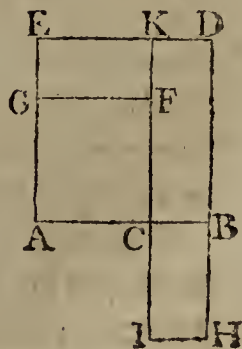
Then the difference of the two fquares AD, AF, is evidently the two rectangles EF, KB. But the rectangles EF, BI are equal, being contained under equal lines; for EK and BH are each equal to AC, and GE is equal to CB, being each equal to the difference between AB and AC, or their equals AE and AG. Therefore the two EF, KB are equal to the two KB, BI, or to the whole KH; and confequently KH is equal to the difference of the fquares AD, AF. But KH is a rectangle contained under DH, or the fum of AB and AC, and KD, or the difference of AB and AC. Therefore the difference of the fquares of AB, AC, is equal to the rectangle under their fum and difference. Q. E. D.

### THEOREM XXXIV.

In any Right-Angled Triangle, the Square of the Hypothenufe, is equal to the Sum of the Squares of the other two Sides.

Let ABC be a right-angled triangle, having the right angle C; then will the fquare of the hypothenufe AB, be equal to the fum of the fquares of the other two fides AC, CB. Or $AB^2 = AC^2 + BC^2$.
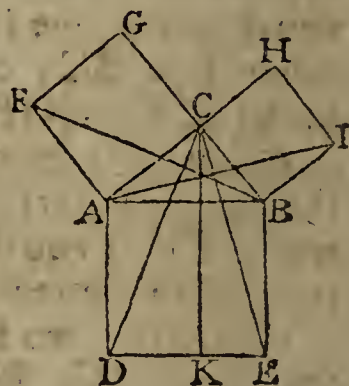
For, on AB defcribe the fquare AE, and on AC, CB the fquares AG, BH; then draw CK parallel to AD or BE; and join AI, BF, CD, CE.

Now, becaufe the line AC meets the two CG, CB, fo as to make two right angles, thefe two make one ftraight line GB (corol. 1, th. 6). And becaufe the angle FAC is equal to the angle DAB, being each a right angle, or the angle of a fquare; to each of thefe equals add the common angle BAC, fo will the whole angle or fum FAB, be equal to the whole angle CAD. But the line FA is equal to the line AC, and the line AB to the line AD, being fides of the fame fquare; fo that the two fides FA, AB, and their included angle FAB, are equal to the two fides CA, AD, and the contained angle CAD, each to each; therefore the whole triangle AFB is equal to the whole triangle ACD (th. 1).

But the fquare AG is double the triangle AFB, on the fame bafe FA, and between the fame parallels FA, GB (th. 26); in like manner, the parallelogram AK is double the triangle ACD, on the fame bafe AD, and between the fame parallels AD, CK. And fince the doubles of equal things,

things, are equal (by ax. 6); therefore the square AG is equal to the parallelogram AK.

In like manner, the other square BH is proved equal to the other parallelogram BK. And consequently the two squares AG and BH together, equal to the two parallelograms AK and BK together, or to the whole square AE. That is, the fum of the two fquares on the two lefs fides, is equal to the fquare on the greatest side. Q. E. D.

*Corol.* Hence, the fquare of either of the two lefs fides, is equal to the difference of the fquares of the hypothenufe and the other fide (ax. 3); or, equal to the rectangle contained under the fum and difference of the faid hypothenufe and other fide (th. 33).

## THEOREM XXXV.

In any Triangle, the Difference of the Squares of the two Sides, is Equal to the Difference of the Squares of the two Lines, or Diftances, included between the Extremes of the Bafe and the Perpendicular.

Let ABC be any triangle, having CD perpendicular to AB; then will the difference of the fquares of AC, BC, be equal to the difference of the fquares of AD, BD; that is, $AC^2 - BC^2 = AD^2 - BD^2$.
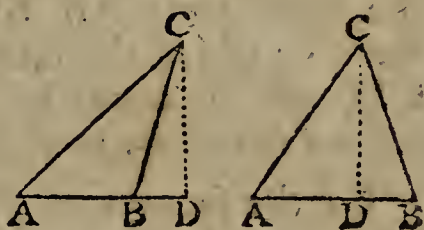
For, fince $AC^2$ is equal to $AD^2 + CD^2$
and $BC^2$ is equal to $BD^2 + CD^2$ }, by th. 34;
Theref. the difference between $AC^2$ and $BC^2$
is equal to the difference between $AD^2 + CD^2$
and $BD^2 + CD^2$
or equal to the difference between $AD^2$ and $BD^2$
by taking away the common fquare $CD^2$. Q. E. D.

*Corol.* The rectangle under the fum and difference of the two fides of any triangle, is equal to the rectangle under the fum and difference of the diftances between the perpendicular and the two extremes of the bafe, or equal to the rectangle under the bafe and the difference or fum of the fegments, according as the perpendicular falls within or without the triangle.

That

That is, $\overline{AC + BC} \cdot \overline{AC - BC} = \overline{AD + BD} \cdot \overline{AD - BD}$

Or, $\overline{AC + BC} \cdot \overline{AC - BC} = AB \cdot \overline{AD - BD}$ in the 2d figure.

And $\overline{AC + BC} \cdot \overline{AC - BC} = AB \cdot \overline{AD + BD}$ in the 1ſt figure.

### THEOREM XXXVI.

In any Obtuſe-angled Triangle, the Square of the Side ſubtending the Obtuſe Angle, is Greater than the Sum of the Squares of the other two Sides, by Twice the Rectangle of the Baſe and the Diſtance of the Perpendicular from the Obtuſe Angle.

Let ABC be a triangle, obtuſe-angled at B, and CD perpendicular to AB; then will the ſquare of AC be greater than the ſquares of AB, BC, by twice the rectangle of AB, BD. That is, $AC^2 = AB^2 + BC^2 + 2AB \cdot BD$. See the 1ſt fig. above, or below.

For, ſince the ſquare of the whole line AD is equal to the ſquares of the parts AB, BD, with twice the rectangle of the ſame parts AB, BD (th. 31); if to each of theſe equals there be added the ſquare of CD, then the ſquares of AD, CD, will be equal to the ſquares of AB, BD, CD, with twice the rectangle of AB, BD (by ax. 2).

But the ſquares of AD, CD are equal to the ſquare of AC; and the ſquares of BD, CD, equal to the ſquare of BC (th. 34); therefore the ſquare of AC is equal to the ſquares of AB, BC, together with twice the rectangle of AB, BD. Q. E. D.

### THEOREM XXXVII.

In any Triangle, the ſquare of the Side ſubtending an Acute Angle, is Leſs than the Squares of the Baſe and the other Side, by Twice the Rectangle of the Baſe and the Diſtance of the Perpendicular from the Acute Angle.

Let ABC be a triangle, having the angle A acute, and CD perpendicular to AB; then will the ſquare of BC, be leſs than the ſquares of AB, AC, by twice the rectangle of AB, AD. That is, $BC^2 = AB^2 + AC^2 - 2AB \cdot AD$.

For,
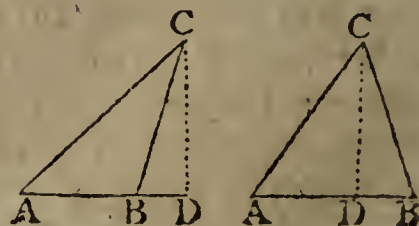
For, in fig. 1, $AC^2$ is $= BC^2 + AB^2 + 2AB \cdot BD$ (th. 36).
To each of these equals add the square of AB,
then is $AB^2 + AC^2 = BC^2 + 2AB^2 + 2AB \cdot BD$ (ax. 2),
or $= BC^2 + 2AB \cdot AD$ (th. 30).

Q. E. D.

Again, in fig. 2, $AC^2$ is $= AD^2 + DC^2$ (th. 34).
And $AB^2 = AD^2 + DB^2 + 2AD \cdot DB$ (th. 31).
Theref. $AB^2 + AC^2 = BD^2 + DC^2 + 2AD^2 + 2AD \cdot DB$ (ax. 2),
or $= BC^2 + 2AD^2 + 2AD \cdot DB$ (th. 34),
or $= BC^2 + 2AB \cdot AD$ (th. 30).     Q. E. D.

## THEOREM XXXVIII.

IN any Triangle, the Double of the Square of a Line drawn from the Vertex to the Middle of the Base, together with Double the Square of the Half Base, is Equal to the Sum of the Squares of the other Two Sides.

Let ABC be a triangle, and CD the line drawn from the vertex to the middle of the base AB, bisecting it into the two equal parts AD, DB: then will the sum of the squares of AC, CB, be equal to twice the sum of the squares of CD, BD; or $AC^2 + CB^2 = 2CD^2 + 2DB^2$.
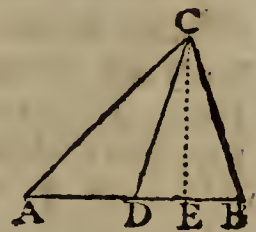
For, let CE be perpendicular to the base AB. Then, since (by th. 36) $AC^2$ exceeds the sum of the two squares $AD^2$ and $CD^2$ (or $BD^2$ and $CD^2$) by the double rectangle $2AD \cdot DE$ (or $2BD \cdot DE$); and since (by th. 37) $BC^2$ is less than the same sum by the said double rectangle; it is manifest that both $AC^2$ and $BC^2$ together, must be equal to that sum twice taken; the excess on the one part making up the defect on the other. Q. E. D.

## THEOREM XXXIX.

IN an Isosceles Triangle, the Square of a Line drawn from the Vertex to any Point in the Base, together with the Rectangle of the Segments of the Base, is Equal to the Square of one of the Equal Sides of the Triangle.

Let ABC be the isosceles triangle, and CD a line drawn from the vertex to any point D in the base: then will the square of AC be equal to the square of CD together with the rectangle of AD and DB. That is, $AC^2 = CD^2 + AD \cdot DB$.

For,

For, let CE bifect the vertical angle: then will it alfo bifect the bafe AB perpendicularly, making AE = EB (corol. 1, th. 3).

But, in the triangle ACD, obtufe-angled at D, the fquare $AC^2$ is $= CD^2 + AD^2 + 2AD \cdot DE$ (th. 36),

  or $= CD^2 + AD \cdot \overline{AD + 2DE}$ (th. 30),

  or $= CD^2 + AD \cdot \overline{AE + DE}$,

  or $= CD^2 + AD \cdot \overline{BE + DE}$,

  or $= CD^2 + AD \cdot DB$.

Q. E. D.

## THEOREM XL.

In any Parallelogram, the two Diagonals Bifect each other, and the Sum of their Squares is Equal to the Sum of the Squares of all the Four Sides of the Parallelogram.

Let ABCD be a parallelogram, whofe diagonals interfect each other in E: then will AE be equal to EC, and BE to ED; and the fum of the fquares of AC, BD, will be equal to the fum of the fquares of AB, BC, CD, DA. That is,
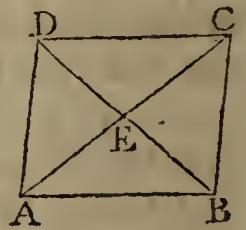
  AE = EC, and BE = ED,

and $AC^2 + BD^2 = AB^2 + BC^2 + CD^2 + DA^2$.

For, the triangles AEB, DEC are equiangular, becaufe they have the oppofite angles at E equal (th. 7). and the two lines AC, BD, meeting the parallels AB, DC, make the angle BAE equal to the angle DCE, and the angle ABE equal to the angle CDE, and the fide AB equal to the fide DC (th. 22); therefore thefe two triangles are identical, and have their correfponding fides equal (th. 2), viz. AE = EC, and BE = ED.

Again, fince AC is bifected in E, the fum of the fquares $AD^2 + DC^2 = 2AE^2 + 2DE^2$ (th. 38).

In like manner, $AB^2 + BC^2 = 2AE^2 + 2BE^2$ or $2DE^2$. Theref. $AB^2 + BC^2 + CD^2 + DA^2 = 4AE^2 + 4DE^2$ (ax. 2).

But, becaufe the fquare of a whole line is equal to four times the fquare of half the line (cor. th. 31), that is, $AC^2 = 4AE^2$, and $BD^2 = 4DE^2$.

Theref. $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2$ (ax. 1).

Q. E. D.

## THEOREM XLI.

IF a Right Line, drawn through or from the Centre of a Circle, Bifect a Chord, it will be Perpendicular to it ; or if it be Perpendicular to the Chord, it will Bifect both the Chord and the Arc of the Chord.

Let AB be any chord in a circle, and CD a line drawn from the centre C to the chord. Then, if the chord is bifected in the point D, CD will be perpendicular to AB.

For, draw the two radii CA, CB. Then the two triangles ACD, BCD, having CA equal to CB (def. 44), CD common, and AD equal to DB (by hyp.), have all the three fides of the one, equal to all the three fides of the other, and fo have their angles alfo equal (th. 5). Hence then, the angle ADC being equal to the angle BDC, thefe angles are right angles, and the line CD is perpendicular to AB (def. 11).

Again, if CD be perpendicular to AB, then will the chord AB be bifected at the point D, or have AD equal to DB, and the arc AEB in the point E, or have AE equal EB.

For, having drawn CA, CB, as before. Then, in the triangle ABC, becaufe the fide CA is equal to the fide CB, their oppofite angles A and B are alfo equal (th. 3). Hence then, in the two triangles ACD, BCD, the angle A is equal to the angle B, and the angles at D are equal (def. 11); therefore their third angles are alfo equal (corol. 1, th 17). And having the fide CD common, they have alfo the fide AD equal to the fide DB (th. 2).

Alfo, fince the angle ACE is equal to the angle BCE, the arc AE which meafures the former (def. 57), is equal to the arc BE which meafures the latter, fince equal angles muft have equal meafures.

*Corol.* Hence a line bifecting any chord at right angles, paffes through the centre of the circle.

## THEOREM XLII.

IF More than Two Equal Right Lines can be drawn from any Point within a Circle to the Circumference, that Point will be the Centre.

Let ABC be a circle, and D a point within it; then if any three lines DA, DB, DC, drawn from the point D to the circumference, be equal to each other, the point D will be the centre.
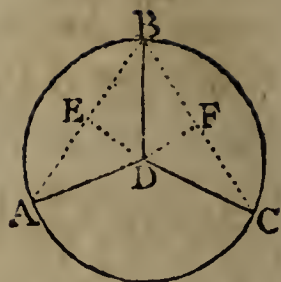
For, draw the chords AB, BC, which let be bifected in the points E, F, and join DE, DF.

Then, the two triangles DAE, DBE, have the fide DA equal to the fide DB by fuppofition, the fide AE equal to the fide EB by hypothefis, and the fide DE common; therefore thefe two triangles are identical, and have the angles at E equal to each other (th. 5); confequently DE is perpendicular to the chord AB (def. 11), and therefore paffes through the centre of the circle (corol. th. 41).

In like manner, it may be fhewn that DF paffes through the centre. Confequently the point D is the centre of the circle, and the three equal lines DA, DB, DC, are radii.

Q. E. D.

## THEOREM XLIII.

IF two Circles touch one another Internally, the Centres of the Circles and the Point of Contact will be all in the Same Right Line.

Let the two circles ABC, ADE touch one another internally in the point A; then will the point A and the centres of thofe circles be all in the fame right line.
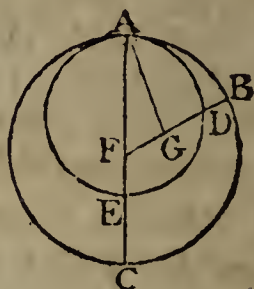
For, let F be the centre of the circle ABC, through which draw the diameter AFC. Then, if the centre of the other circle can be out of this line AC, let it be fuppofed in fome other point as G; through which draw the line FG cutting the two circles in B and D.

Now, in the triangle AFG, the fum of the two fides FG, GA is greater than the third fide AF (th. 10), or greater than its equal radius FB. From each of thefe take away the common part FG, and the remainder GA will be greater than the remainder GB. But the point G being fuppofed the centre of the inner circle, its two radii GA, GD are equal to one another; confequently GD will alfo be greater than GB. But ADE being the inner circle, GD is neceffarily

lefs

lefs than GB. So that GD is both greater and lefs than GB ; which is abfurd. Confequently the centre G cannot be out of the line AFC. Q. E. D.

## THEOREM XLIV.

If two Circles Touch one another Externally, the Centres of the Circles and the Point of Contact will be all in the Same Right Line.

Let the two circles ABC, ADE touch one another externally at the point A ; then will the point of contact A and the centres of the two circles be all in the fame right line.

For, let F be the centre of the circle ABC, through which draw the diameter AFC, and produce it to the other circle at E. Then, if the centre of the circle ADE can be out of the line FE, let it, if poffible, be fuppofed in fome other point G ; and draw the lines AG, FBDG, cutting the two circles in B and D.
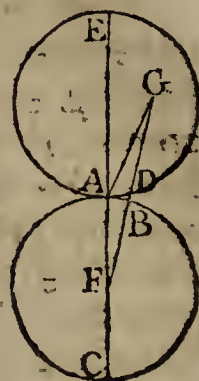
Then, in the triangle AFG, the fum of the two fides AF, AG is greater than the third fide FG (th. 10). But, F and G being the centres of the two circles, the two radii GA, GD are equal, as are alfo the two radii AF, FB. Hence the fum of GA, AF is equal to the fum of GD, BF ; and therefore this latter fum alfo GD, BF is greater than GF, which is abfurd. Confequently the centre G cannot be out of the line FE. Q. E. D.

## THEOREM XLV.

Any two Chords in a Circle, which are Equally Diftant from the Centre, are Equal to each other ; or if they be Equal to each other, they will be Equally Diftant from the Centre.

Let AB, CD be any two chords at equal diftances from the centre G ; then will thefe two chords AB, CD be equal to each other.
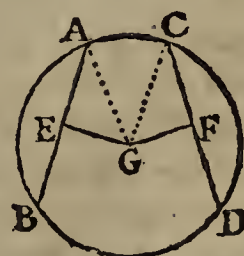
For, draw the two radii GA, GC, and the two perpendiculars GE, GF, which are the equal diftances from the centre G. Then, the two right-angled triangles GAE, GCF, having the fide GA equal the fide GC, and the fide GE equal the fide GF, the difference of the fquares $GA^2$, $GE^2$ will be equal the difference of the fquares

ſquares GC², GF². But AE² is equal to
the difference of the ſquares GA², GE²,
and CF² equal the difference of the ſquares
GC², GF² (corol. th. 34); therefore the
ſquare AE² is equal the ſquare CF², and
the line AE equal the line CF. But AB is
the double of AE, and CD is the double of CF (th. 41);
therefore AB is equal CD (by ax. 6). Q. E. D.
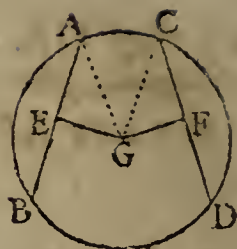
Again, if the chord AB be equal to the chord CD; then
will their diſtances from the centre, GE, GF, alſo be equal
to each other.

For, ſince AB is equal CD by ſuppoſition, the half AE is
equal the half CF. Alſo the radii GA, GC being equal,
the difference of the ſquares GA², GE² will be equal the
difference of the ſquares GC², GF²; that is, the ſquare GE²
equal the ſquare GF² (corol. th. 34), and conſequently the
diſtance GE equal the diſtance GF. Q. E. D.

*Corol.* Hence, if two right-angled triangles having equal
hypothenuſes, have two other ſides alſo equal; then will
the third ſides be equal, and the two triangles identical, or
equal in all reſpects.

## THEOREM XLVI.

A Right Line Perpendicular to the Extremity of a Radius,
is a Tangent to the Circle.

LET the line ADB be perpendicular to the
radius CD of a circle; then ſhall AB touch
the circle in the point D only.
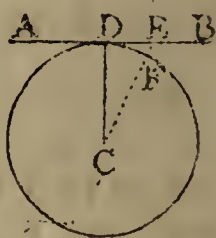
For, from any other point E in the line
AB, draw CFE to the centre, meeting the
circle in F.

Then, becauſe the angle D, of the triangle
CDE, is right-angled at D, the angle at E is acute (th. 17.
cor. 3). and conſequently leſs than the angle D. But the
greater ſide is always oppoſite to the greater angle (th. 9);
therefore the ſide CE is greater than the ſide CD, or greater
than its equal CF. Hence the point E is without a circle;
and the ſame for every other point in the line AB. Conſe-
quently the whole line is without the circle, and meets it in
the point D only.

*Corol.* Hence it follows, that a perpendicular is the
ſhorteſt line which can be drawn from any point to a given
line; ſince the perpendicular CD is ſhorter than any other
line which can be drawn from the point C to the line AB,
the ſame as theorem 21.

THE

## THEOREM XLVII.

IF a Right Line be a Tangent to a Circle; a Radius drawn to the Point of Contact will be Perpendicular to the Tangent.

Let the line AB touch the circumference of a circle at the point D; then will the radius CD be perpendicular to the tangent AB. [See the laft figure.]

For, the line AB being wholly without the circumference except at the point D, every other line, as CE, drawn from the centre C to the line AB, muft pafs out of the circle to arrive at this line. The line CD is therefore the fhorteft that can be drawn from the point C to the line AB, and confequently (th. 21), is perpendicular to that line.

*Corol.* Hence, converfely, a line drawn perpendicular to a tangent, at the point of contact, paffes through the centre of the circle.

## THEOREM XLVIII.

The Angle formed by a Tangent and Chord is Meafured by Half the Arc of that Chord.

LET AB be a tangent to a circle, and CD a chord drawn from the point of contact C: then is the angle BCD meafured by half the arc CFD, and the angle ACD meafured by half the arc CGD.
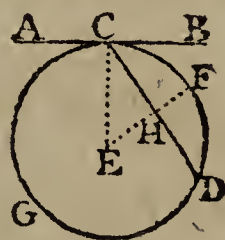
For, draw the radius EC to the point of contact, and the radius EF perpendicular to the chord at H.

Then, the radius EF, being perpendicular to the chord CD, bifects the arc CFD (th. 41). Therefore CF is half the arc CFD.

In the triangle CEH, the angle H being a right angle, the fum of the two remaining angles E and C is equal to a right angle (corol. 3, th. 17), which is equal to the angle BCE, becaufe the radius CE is perpendicular to the tangent. From each of thefe equals take away the common part or angle C, and there remains the angle E equal to the angle BCD. But the angle E is meafured by the arc CF (def. 57), which is the half of CFD; therefore the equal angle BCD muft alfo have the fame meafure, namely, half the arc CFD of the chord CD.

Again,

Again, the line GEF, being perpendicular to the chord CD, bisects the arc CGD (th. 41). Therefore CG is half the arc CGD. Now, since the line CE, meeting FG, makes the sum of the two angles at E equal to two right angles (th. 6), and the line CD makes with AB the sum of the two angles at C equal to two right angles; if from these two equal sums there be taken away the parts or angles CEH and BCH, which have been proved equal, there remains the angle CEG equal to the angle ACH. But the former of these, CEG, being an angle at the centre, is measured by the arc CG (def. 57), consequently the equal angle ACD must also have the same measure CG, which is half the arc CGD of the chord CD.  Q. E. D.
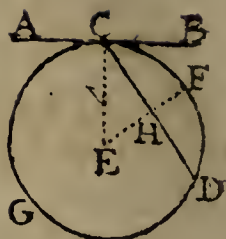
*Corol.* 1. The sum of two right angles is measured by half the circumference. For the two angles BCD, ACD, which make up two right angles, are measured by the arcs CF, CG, which make up half the circumference, FG being a diameter.

*Corol.* 2. Hence also one right angle must have for its measure a quarter of the circumference, or 90 degrees.

## THEOREM XLIX.

An Angle at the Circumference of a Circle, is Measured by Half the Arc that subtends it.

LET BAC be an angle at the circumference; it has for its measure, half the arc BC which subtends it.

For, suppose the tangent DE passing through the point of contact A. Then, the angle DAC being measured by half the arc ABC, and the angle DAB by half the arc AB (th. 48); it follows, by equal subtraction, that the difference, or angle BAC, must be measured by half the arc BC, which it stands upon.  Q. E. D.

## THEOREM L.

All Angles in the Same Segment of a Circle, or Standing on the Same Arc, are Equal to each other.

LET ACB, ADB be two angles in the fame fegment ACDB, or, which is the fame thing, ftanding on the fame arc AEB; then will the angle ACB be equal to the angle ADB.

For each of thefe angles is meafured by half the arc AEB; and thus, having equal meafures, they are equal to each other (ax. 11).

## THEOREM LI.

An Angle at the Centre of a Circle is Double the Angle at the Circumference, when both of them ftand on the Same Arc.

Let ACB be an angle at the centre C, and ADB an angle at the circumference, both ftanding on the fame arc or fame chord AB: then will the angle C be double of the angle D, or the angle D equal to half the angle C.
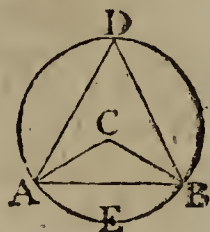
For the angle at the centre C is meafured by the whole arc AEB (def. 57); and the angle at the circumference D is meafured by half the fame arc AEB (th. 49); therefore the angle D is only half the angle C, or the angle C double the angle D.

## THEOREM LII.

An Angle in a Semicircle, is a Right Angle.

IF ABC, or ADC be a Semicircle; then any angle ABC in that femicircle, is a right angle.
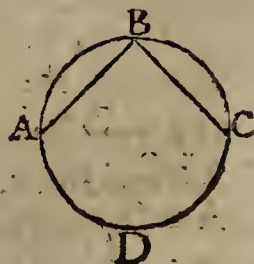
For, the angle B, at the circumference, is meafured by half the arc ADC (th. 49), that is, by a quadrant of the circumference. But a quadrant is the meafure of a right angle (corol. 4, th. 6; or corol. 2, th. 48). Therefore the angle B is a right angle.

### THEOREM LIII.

THE Angle formed by a Tangent to a Circle and a Chord drawn from the Point of Contact, is Equal to the Angle in the Alternate Segment.

If AB be a tangent, AC a chord, and D any angle in the alternate fegment ADC; then will the angle D be equal to the angle BAC made by the tangent and chord of the arc AEC.
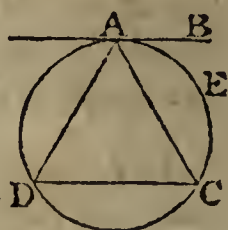
For the angle D, at the circumference, is meafured by half the arc AEC (th. 49); and the angle BAC, made by the tangent and chord, is alfo meafured by the fame half arc AEC (th. 48); therefore thefe two angles are equal (ax. 11).

### THEOREM LIV.

The Sum of any Two Oppofite Angles of a Quadrangle In-fcribed in a Circle, is Equal to Two Right Angles.

LET ABCD be any quadrilateral infcribed in a circle; then fhall the fum of the two oppofite angles A and C, or B and D, be equal to two right angles.
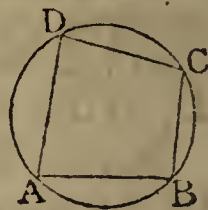
For the angle A is meafured by half the arc DCB, which it ftands upon, and the angle C by half the arc DAB (th. 49); therefore the fum of the two angles A and C is meafured by half the fum of thefe two arcs, that is, by half the circum-ference. But half the circumference is the meafure of two right angles (corol. 4, th. 6); therefore the fum of the two oppofite angles A and C is equal to two right angles. In like manner, it is fhewn that the fum of the other two oppo-fite angles, D and B, is equal to two right angles. Q. E. D.

### THEOREM LV.

IF any Side of a Quadrangle, Infcribed in a Circle, be Produced out, the Outward Angle will be Equal to the In-ward Oppofite Angle.

IF the fide AB, of the quadrilateral ABCD, infcribed in a circle, be produced to E; the outward angle DAE will be equal to the inward oppofite angle C.

For,
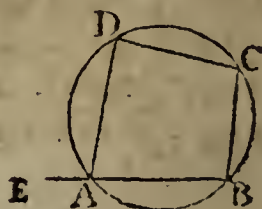
For, the fum of the two adjacent angles DAE and DAB
is equal to two right angles (th. 6) ; and the fum of the two
oppofite angles C and DAB is alfo equal to two right angles
(th. 54) ; therefore the fum of the two angles DAE and DAB
is equal to the fum of the two C and DAB (ax. 1).  From
each of thefe equals taking away the common angle DAB,
there remains the angle DAE equal the angle C.  Q. E. D.

## THEOREM LVI.

Two Parallel Chords Intercept Equal Arcs.

LET the two chords AB, CD be parallel;
then will the arcs AC, BD be equal; or
AC = BD.
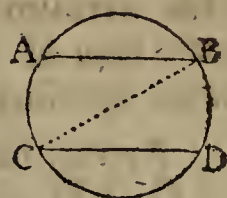
For, draw the line BC.  Then becaufe
the lines AB, CD are parallel, the alternate
angles B and C are equal (th. 12).  But
the angle at the circumference B, is meafured by half the
arc AC (th. 49) ; and the other angle at the circumference
C is meafured by half the arc BD : the halves of the arcs
AC, BD, and confequently the arcs themfelves, are there-
fore equal.  Q. E. D.

## THEOREM LVII.

If a Tangent and Chord be Parallel to each other, they In-
tercept Equal Arcs.

LET the tangent ABC be parallel to the
chord DE ; then are the arcs BD, BE
equal ; that is, BD = BE.
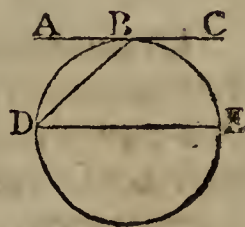
For, draw the chord BD.  Then, becaufe
the lines AB, DE are parallel, the alter-
nate angles D and B are equal (th. 12).
But the angle B, formed by a tangent and a chord, is mea-
fured by half the arc BD (th. 48) ; and the angle at the
circumference D is meafured by half the arc BE (th. 49) ;
the arcs BD, BE are therefore equal.  Q. E. D.

## THEOREM LVIII.

THE Angle formed, Within a Circle, by the Interfection of two Chords, is Meafured by Half the Sum of the Two Arcs Intercepted by thofe Chords.

Let the two chords AB, CD interfect at the point E; the angle AEC, or DEB, is meafured by half the fum of two arcs AC, DB.
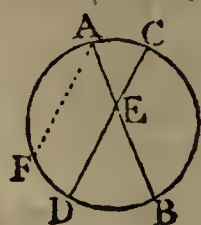
For, draw the chord AF parallel to CD. Then, becaufe the lines AF, CD are parallel, and AB cuts them, the angles on the fame fide A and DEB are equal (th. 14). But the angle at the circumference A is meafured by half the arc BF, or of the fum of FD and DB (th. 49); therefore the angle E is alfo meafured by half the fum of FD and DB.

Again, becaufe the chords AF, CD are parallel, the arcs AC, FD are equal (th. 56); therefore the fum of the two arcs AC, DB is equal to the fum of the two FD, DB; and confequently the angle E, which is meafured by half the latter fum, is alfo meafured by half the former.   Q. E. D.

## THEOREM LIX.

The Angle formed, Without a Circle, by two Secants, is Meafured by Half the Difference of the Intercepted Arcs.

LET the angle E be formed by two fecants AB and CD, this angle is meafured by half the difference of the two arcs AC, DB, intercepted by the two fecants.
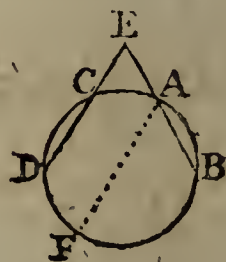
Draw the chord AF parallel to CD. Then, becaufe the lines AF, CD are parallel, and AB cuts them, the angles on the fame fide A and BED are equal (th. 14). But the angle A, at the circumference, is meafured by half the arc BF (th. 49), or of the difference of DF and DB: therefore the equal angle E is alfo meafured by half the difference of DF, DB.

Again, becaufe the chords AF, CD are parallel, the arcs AC, FD are equal (th. 56); therefore the difference of the two arcs AC, DB is equal to the difference of the two DF, DB. Confequently the angle E, which is meafured by half the latter difference, is alfo meafured by half the former.

Q. E. D.

THE

## THEOREM LX.

The Angle formed by two Tangents, is Meafured by Half the Difference of the two Intercepted Arcs.

LET EB, ED be two tangents to a circle at the points A, C; then the angle E is meafured by half the difference of the two arcs CFA, CGA.
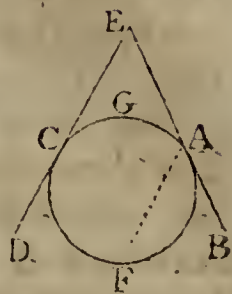
For, draw the chord AF parallel to CD. Then, becaufe the lines AF, CD are parallel, and AB meets them, the angles on the fame fide A and E are equal (th. 14). But the angle A, formed by the chord AF and tangent AB, is meafured by half the arc AF (th. 48); therefore the equal angle E is alfo meafured by half the fame arc AF, or half the difference of the arcs CFA and CF.

Again, becaufe the tangent ED and chord AF are parallel, the intercepted arcs CGA, CF, are equal (th. 57); the arc AF therefore, which is the difference of CFA and CF, is alfo the difference of CFA and CGA; confequently the angle E, which is meafured by half the former, is alfo meafured by half the latter.

*Corol.* In like manner it is proved, that the angle E, formed by a tangent ECD and a fecant EAB, is meafured by half the difference of the two intercepted arcs CA and CFB.
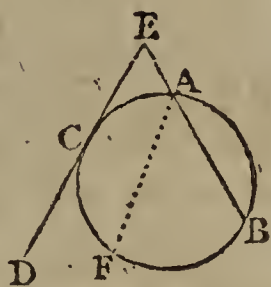
## THEOREM LXI.

IF two Lines, meeting a Circle each in two Points, Cut one other, either Within it or Without it; the Rectangle of the Parts of the one, will be Equal to the Rectangle of the Parts of the other; the Parts of each being meafured from the point of meeting to the two interfections with the circumference.

Let the two chords AB, CD meet each other in E; the rectangle of AE, EB, is equal to the rectangle of CE, ED. Or, AE . EB = CE . ED.

For, through the point E draw the diameter FG; also from the centre H, draw the radius DH, and HI perpendicular to CD.

Then, since DEH is a triangle, and the perp. HI bisects the chord CD (th. 41), the line CE is equal to the difference of the segments DI, EI, the sum of them being DE. Also, because H is the centre of the circle, and the radii DH, FH, GH are all equal, the line EG is equal to the sum of the sides DH, HE ; and EF is equal to their difference.

But the rectangle of the sum and difference of the two sides of a triangle, is equal to the rectangle of the sum and difference of the segments of the base (corol. th. 35) ; therefore the rectangle of FE, EG, is equal to the rectangle of CE, ED. In the same manner it is proved, that the rectangle of FE, EG, is equal to the rectangle of AE, EB. Consequently the rectangle of AE, EB, is also equal to the rectangle of CE, ED. Q. E. D.

*Corol.* 1. When one of the lines in the second case, as DE, by revolving about the point E, comes into the position of the tangent EC or ED, the two points C and D running into one ; then the rectangle CE, ED becomes the square of CE, because CE and DE are then equal. Consequently the rectangle of the parts of the secant, AE . EB, is equal to the square of the tangent $CE^2$,

*Corol.* 2. Hence both the tangents EC, EF, drawn from the same point E, are equal ; since the square of each is equal to the same rectangle or quantity.

In Equiangular Triangles, the Rectangles of the Corresponding or Like Sides, taken alternately, are Equal.

LET

3

Let ABC, DEF be two equiangular triangles, having the angle A = the angle D, the angle B = the angle E, and the angle C = the angle F; alſo the like ſides AB, DE, and AC, DF, being thoſe oppoſite the equal angles: then will the rectangle of AB, DF, be equal to the rectangle of AC, DE.
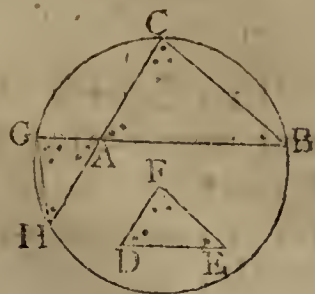
In BA produced take AG equal DF; then through the three points B, C, G, conceive a circle BCGH to be deſcribed, meeting CA produced at H, and join GH.

Then the angle G is equal to the angle C on the ſame arc BH, and the angle H equal to the angle B on the ſame arc CG (th. 50); alſo the oppoſite angles at A are equal (th. 7): therefore the triangle AGH is equiangular to the triangle ACB, and conſequently to the triangle DFE alſo. But the two like ſides AG, DF are alſo equal by ſuppoſition; conſequently the two triangles AGH, DFE are indentical (th. 2), having the two ſides AG, AH equal to the two DF, DE, each to each.

But the rectangle GA . AB is equal to the rectangle HA . AC (th. 61), and conſequently the rectangle DF . AB equal the rectangle DE . AC. Q. E. D.

## THEOREM LXIII.

THE Rectangle of the two Sides of any Triangle, is Equal to the Rectangle of the Perpendicular on the third Side and the Diameter of the Circumſcribing Circle.

Let CD be the perpendicular, and CE the diameter of the circle about the triangle ABC: then the rectangle AC . CB is = the rectangle CD . CE.
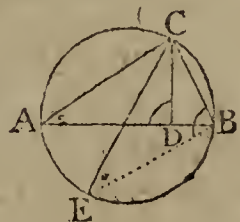
For, join BE: then in the two triangles ACD, ECB, the angles A and E are equal, ſtanding on the ſame arc BC (th. 50); alſo the right angle D is equal the angle B, which is alſo a right angle, being in a ſemicircle (th. 52): therefore theſe two triangles have alſo their third angles equal, and are equiangular. Hence, AC, CE, and CD, CB being like ſides, ſubtending the equal angles, the rectangle AC . CB, of the firſt and laſt of them, s equal to the rectangle CE . CD of the other two (th. 62).

THE-

## THEOREM LXIV.

THE Square of a Line Bifecting any Angle of a Triangle, and terminating in the Oppofite Side, together with the Rectangle of the two Segments of that Side, is Equal to the Rectangle of the two Sides including the Bifected Angle.

Let CD bifect the angle C of the triangle ABC; then the fquare $CD^2$ + the rectangle AD . DB is = the rectangle AC . CB.
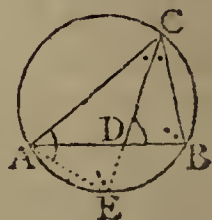
For, let CD be produced to meet the circumfcribing circle at E, and join AE.

Then the two triangles ACE, BCD are equiangular : for the angles at C are equal by fuppofition, and the angles B and E are equal, ftanding on the fame arc AC (th. 50) ; confequently the third angles at A and D are equal (corol. 1, th. 17): alfo AC, CD, and CE, CB are like or corresponding fides, being oppofite to equal angles: therefore the rectangle AC . CB is = the rectangle CD . CE (th. 62). But the rectangle CD . CE is = $CD^2$ + the rectangle CD . DE (th. 30), and therefore alfo the rectangle AC . CB is = $CD^2$ + CD . DE, or equal $CD^2$ + AD . DB, fince CD . DE is = AD . DB (th. 61).

Q. E. D.

## THEOREM LXV.

THE Rectangle of the two Diagonals of any Quadrangle Infcribed in a Circle, is Equal to the Sum cf the two Rectangles of the Oppofite Sides.

Let ABCD be any quadrilateral infcribed in a circle, and AC, BD its two diagonals : then the rectangle AC . BD is = the rectangle AB . DC + the rectangle AD . BC.

For, let CE be drawn making the angle BCE equal the angle DCA. Then the two triangles ACD, BCE are equiangular; for the angles A and B are equal, ftanding on the fame arc DC ; and the angles DCA, BCE are equal by fuppofition ; confequently the third angles ADC, BEC are equal : alfo, AC, BC, and AD, BE are like or corresponding fides, being oppofite the equal angles : therefore the rect. AC . BE is = AD . BC (th. 62).

Again,

Again, the two triangles ABC, DEC are equiangular : for the angles BAC, BDC are equal, ſtanding on the ſame arc BC ; and the angle DCE is equal the angle BCA, by adding the common angle ACE to the two equal angles DCA, BCE ; therefore the third angles E and ABC are alſo equal : but AC, DC, and AB, DE are the like ſides: therefore the rectangle AC . DE is = the rectangle AB . DC (th. 62).

Hence, by equal additions, the ſum of the rectangles AC . BE + AC . DE is = AD . BC + AB . DC. But the ſum of the rectangles AC . BE + AC . DE is = the rectangle AC . BD (th. 30): therefore the rectangle AC . BD is = the rectangle AD . BC + AB . DC (ax. 1). Q. E. D.

# OF RATIOS AND PROPORTIONS.

## DEFINITIONS.

DEF. 76. RATIO is the proportion or relation which one magnitude bears to another magnitude of the ſame kind, with reſpect to quantity.

*Note.* The meaſure, or quantity, of a ratio, is conceived, by conſidering what part or parts the leading quantity, called the Antecedent, is of the other, called the Conſequent ; or what part or parts the number expreſſing the quantity of the former, is of the number denoting in like manner the latter. So, the ratio of a quantity expreſſed by the number 2, to a like quantity expreſſed by the number 6, is denoted by 6 divided by 2, or $\frac{6}{2}$ or 3 ; the number 2 being 3 times contained in 6, or the third part of it. In like manner, the ratio of the quantity 3 to 6, is meaſured by $\frac{6}{3}$ or 2 ; the ratio of 4 to 6 is $\frac{6}{4}$ or $1\frac{1}{2}$ ; that of 6 to 4 is $\frac{4}{6}$ or $\frac{2}{3}$ ; &c.

77. Proportion is an equality of ratios Thus,

78. Three quantities are ſaid to be Proportional, when the ratio of the firſt to the ſecond is equal to the ratio of the ſecond to the third. As of the three quantities A (2), B (4), C (8): where $\frac{4}{2} = \frac{8}{4} = 2$, both the ſame ratio.

79. Four quantities are ſaid to be Proportional, when the ratio of the firſt to the ſecond, is the ſame as the ratio of the third to the fourth. As of the four A (2), B (4), C (5), D (10) ; where $\frac{4}{2} = \frac{10}{5} = 2$, both the ſame ratio.

*Note.*

*Note.* To denote that four-quantities, A, B, C, D, are proportional, they are ufually ftated or placed thus, A : B :: C : D; and read thus, as A is to B fo is C to D. But when three quantities are proportional, the middle one is repeated, and they are written thus, A : B :: B : C.

80. Of three proportional quantities, the middle one is faid to be a Mean Proportional between the other two; and the laft, a Third Proportional to the firft and fecond.

81. Of four proportional quantities, the laft is faid to be a Fourth Proportional to the other three, taken in order.

82. Quantities are faid to be Continually Proportional, or in Continued Proportion, when the ratio is the fame between every two adjacent terms, viz. when the firft is to the fecond, as the fecond to the third, as the third to the fourth, as the fourth to the fifth and fo on, all in the fame common ratio. As in the quantities 1, 2, 4, 8, 16, &c; where the common ratio is equal to 2.

83. In a feries or rank of quantities continually proportional, the ratio of the firft and third is faid to be Duplicate to that of the firft and fecond; and the ratio of the firft and fourth, Triplicate to that of the firft and fecond; and fo on.

84. Of any number of quantities, A, B, C, D, the ratio of the firft A, to the laft D, is faid to be Compounded of the ratios of the firft to the fecond, of the fecond to the third, and fo on to the laft.

85. Inverfe ratio is, when the antecedent is made the confequent, and the confequent the antecedent.—Thus, if 1 : 2 :: 3 : 6; then inverfely, 2 : 1 :: 6 : 3.

86. Alternate proportion is, when antecedent is compared with antecedent, and confequent with confequent —As, if 1 : 2 :: 3 : 6; then, by alternation, or permutation, it will be 1 : 3 :: 2 : 6.

87. Compounded ratio is, when the fum of the antecedent and confequent is compared, either with the confequent, or with the antecedent —Thus, if 1 : 2 :: 3 : 6, then by compofition, 1 + 2 : 1 :: 3 + 6 : 3, and 1 + 2 : 2 :: 3 + 6 : 6.

88. Divided ratio is, when the difference of the antecedent and confequent is compared, either with the antecedent or with the confequent.—Thus, if 1 : 2 :: 3 : 6, then, by divifion, 2 — 1 : 1 :: 6 — 3 : 3, and 2 — 1 : 2 :: 6 — 3 : 6.

THE-

## THEOREM LXVI.

Equimultiples of any two Quantities are in the Same Ratio as the Quantities themselves.

Let A and B be any two quantities, and $m$A, $m$B, any equimultiples of them, $m$ being any number whatever: then will $m$A and $m$B have the same ratio as A and B, or A : B :: $m$A : $m$B.

For $\dfrac{m\text{B}}{m\text{A}} = \dfrac{\text{B}}{\text{A}}$, the same ratio.

*Corol* Hence, like parts of quantities have the same ratio as the wholes ; becaufe the wholes are equimultiples of the like parts, or A and B are like parts of $m$A and $m$B.

## THEOREM LXVII.

If Four Quantities, of the Same Kind, be Proportionals ; they will be in Proportion by Alternation or Permutation, or the Antecedents will have the Same Ratio as the Confequents.

Let A : B :: $m$A : $m$B ; then will A : $m$A :: B : $m$B.

For $\dfrac{m\text{A}}{\text{A}} = m$, and $\dfrac{m\text{B}}{\text{B}} = m$, both the same ratio.

## THEOREM LXVIII.

If Four Quantities be Proportional ; they will be in Proportion by Inverfion, or Inverfely.

Let A : B :: $m$A : $m$B ; then will B : A :: $m$B : $m$A.

For $\dfrac{m\text{A}}{m\text{B}} = \dfrac{\text{A}}{\text{B}}$, both the same ratio.

## THEOREM LXIX.

If Four Quantities be Proportional ; they will be in Proportion by Compofition and Divifion.

Let A : B :: $m$A : $m$B ;

then will B ± A : A :: $m$B ± $m$A : $m$A,

and B ± A : B :: $m$B ± $m$A : $m$B.

For, $\dfrac{m\text{A}}{m\text{B} \pm m\text{A}} = \dfrac{\text{A}}{\text{B} \pm \text{A}}$; and $\dfrac{m\text{B}}{m\text{B} \pm m\text{A}} = \dfrac{\text{B}}{\text{B} \pm \text{A}}$.

*Corol.*

*Corol.* It appears from hence, that the Sum of the Greateſt and Leaſt of four proportional quantities, of the ſame kind, exceeds the Sum of the Two Means. For, ſince — — — A : A + B :: $m$A : $m$A + $m$B, where A is the leaſt, and $m$A + $m$B the greateſt; then $\overline{m + 1}$ . A + $m$B, the ſum of the greateſt and leaſt, exceeds $\overline{m + 1}$ .. A + B the ſum of the two means.

### THEOREM LXX.

I$_F$, of Four Proportional Quantities, there be taken any Equimultiples whatever of the two Antecedents, and any Equimultiples whatever of the two Conſequents; the quantities reſulting will ſtill be proportional.

Let A : B :: $m$A : $m$B; alſo, let $p$A and $pm$A be any equimultiples of the two antecedents, and $q$B and $qm$B any equimultiples of the two conſequents; then will — — — $p$A : $q$B :: $pm$A : $qm$B.

For $\dfrac{qm\text{B}}{pm\text{A}} = \dfrac{q\text{B}}{p\text{A}}$, both the ſame ratio.

### THEOREM LXXI.

I$_F$ there be Four Proportional Quantities, and the two Conſequents be either Augmented or Diminiſhed by Quantities that have the Same Ratio as the reſpective Antecedents; the Reſults and the Antecedents will ſtill be Proportionals.

Let A : B :: $m$A : $m$B, and $n$A and $nm$A any two quantities having the ſame ratio as the two antecedents; then will A : B $\pm$ $n$A :: $m$A : $m$B $\pm$ $nm$A.

For $\dfrac{m\text{B} \pm nm\text{A}}{m\text{A}} = \dfrac{\text{B} \pm n\text{A}}{\text{A}}$, both the ſame ratio.

### THEOREM LXXII.

I$_F$ any Number of Quantities be Proportional; either of the Antecedents will be to its Conſequent, as the Sum of all the Antecedents, is to the Sum of all the Conſequents.

Let A : B :: $m$A : $m$B :: $n$A : $n$B, &c; then will — — — A : B :: A + $m$A + $n$A :: B + $m$B + $n$B, &c.

For $\dfrac{\text{B} + m\text{B} + n\text{B}}{\text{A} + m\text{A} + n\text{A}} = \dfrac{\text{B}}{\text{A}}$, the ſame ratio.

## THEOREM LXXIII.

IF a Whole Magnitude be to a Whole, as a Quantity taken from the first, is to a Quantity taken from the other; the Remainder will be to the Remainder, as the whole to the whole.

Let $A : B :: \frac{m}{n} A : \frac{m}{n} B$;

then will $A : B :: A - \frac{m}{n} A : B - \frac{m}{n} B$.

For $\dfrac{B - \frac{m}{n} B}{A - \frac{m}{n} A} = \dfrac{B}{A}$, both the fame ratio.

## THEOREM LXXIV.

IF Quantities be Proportional; their Squares, or Cubes, or any Like Powers, or Roots, of them, will alfo be Proportional.

Let $A : B :: mA : mB$; then will $A^n : B^n :: m^n A^n : m^n B^n$.

For $\dfrac{m^n B^n}{m^n A^n} = \dfrac{B^n}{A^n}$, both the fame ratio.

## THEOREM LXXV.

IF there be Two Sets of Proportionals: then the Products or Rectangles of the Correfponding Terms will alfo be Proportional.

Let $A : B :: mA : mB$,
and $C : D :: nC : nD$;
then will $AC : BD :: mnAC : mnBD$.

For $\dfrac{mnBD}{mnAC} = \dfrac{BD}{AC}$, both the fame ratio.

## THEOREM LXXVI.

IF Four Quantities be Proportional; the Rectangle or Product of the two Extremes, will be Equal to the Rectangle or Product of the two Means. And the converfe.

Let $A : B :: mA : mB$;
then is $A \times mB = B \times mA = mAB$, as is evident.

## THEOREM LXXVII.

IF Three Quantities be Continual Proportionals; the Rectangle or Product of the two Extremes, will be equal to the Square of the Mean. And the converse.

Let A, $m$A, $m^2$A be three proportionals,
or  A : $m$A :: $m$A : $m^2$A ;
then is A $\times$ $m^2$A $= m^2$A$^2$, as is evident.

## THEOREM LXXVIII.

IF any Number of Quantities be Continual Proportionals; the Ratio of the First to the Third, will be duplicate or the Square of the Ratio of the First and Second ; the Ratio of the First and Fourth will be triplicate or the cube of that of the First and Second ; and so on.

Let A, $m$A, $m^2$A, $m^3$A, &c, be proportionals ;

then is $\dfrac{m\text{A}}{\text{A}} = m$ ; but $\dfrac{m^2\text{A}}{\text{A}} = m^2$ ; and $\dfrac{m^3\text{A}}{\text{A}} = m^3$ ; &c.

## THEOREM LXXIX.

TRIANGLES, and also Parallelograms, having the Same Altitude, or that are between the Same Parallels, are to one another in the same Ratio as their Bases.

Let the two triangles ADC, DFF, have the same altitude, or be between the same parallels AE, CF ; then is the surface of the triangle ADC to the surface of the triangle DEF. as the base AD is to the base DE. Or, AD : DE :: the triangle ADC : the triangle DEF.
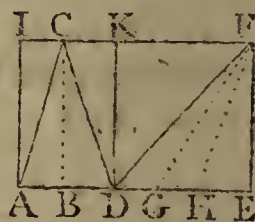
For, let the base AD be to the base DE, as any one number $m$ (2), to any other number $n$ (3) ; and divide the respective bases into those parts, AB, BD, DG, GH, HE, all equal to one another ; and from the points of division draw the lines BC, FG, FH, to the vertexes C and F. Then will these lines divide the triangles ADC, DEF into the same number of parts as their bases, each equal to the triangle ABC, because those triangular parts have equal bases and altitude (corol. 2, th. 25) ; namely, the triangle ABC equal to each of the triangles BDC, DFG, GFH, HFE. So that the triangle ADC is to the triangle DFE, as the number of
parts

parts *m* (2) of the former, to the number *n* (3) of the latter, that is, as the bafe AD to the bafe DE (def. 79).

In like manner, the parallelogram ADKI is to the parallelogram DEFK, as the bafe AD is to the bafe DE ; each of thefe having the fame ratio as the number of their parts, *m* to *n*.   Q. E. D.

## THEOREM LXXX.

Triangles and Parallelograms, having Equal Bafes, are to each other as their Altitudes.

LET AEC, BEF be two triangles, having the equal bafes AB, BE, and whofe altitudes are the perpendiculars CG, FH ; then will the triangle ABC: the triangle BEF :: CG : FH.

For, let BK be perpendicular to AB, and equal to CG ; in which let there be taken BL = FH ; drawing AK and AL.
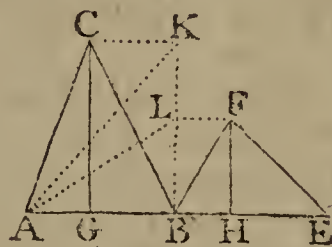
Then, triangles of equal bafes and heights being equal (corol. 2, th. 25), the triangle ABK is = ABC, and the triangle ABL = BEF. But, confidering now ABK, ABL as two triangles on the bafes BK, BL, and having the fame altitude AB, thefe will be as their bafes (th. 79), namely, the triangle ABK : the triangle ABL :: BK : BL.

But the triangle ABK = ABC, and the triangle ABL = BEF, alfo BK = CG, and BL = FH.
Theref. the triangle ABC : triangle BEF :: CG : FH.

And fince parallelograms are the doubles of thefe triangles, having the fame bafes and altitudes, they will likewife have to each other the fame ratio as their altitudes.   Q. E. D.

*Corol.* Since, by this theorem, triangles and parallelograms, when their bafes are equal, are to each other as their altitudes ; and by the foregoing one, when their altitudes are equal, they are to each other as their bafes ; therefore univerfally, when neither are equal, they are to each other in the compound ratio, or as the rectangle or product of their bafes and altitudes.

## THEOREM LXXXI.

IF Four Lines be Proportional ; the Rectangle of the Extremes will be Equal to the Rectangle of the Means. And, converfely, if the Rectangle of the Extremes, of four Lines, be Equal to the Rectangle of the Means, the Four Lines, taken alternately, will be Proportional.

Let the four lines A, B, C, D be proportionals, or A : B :: C : D ; then will the rectangle of A and D be equal to the rectangle of B and C ; or the rectangle A.D = B.D
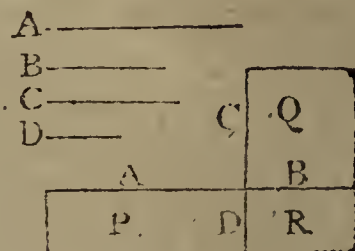
For, let the four lines be placed, with their four extremities meeting in a common point, forming at the point four right angles; and draw lines parallel to them to complete the rectangles P, Q, R, where P is the rectangle of A and D, Q the rectangle of B and C, and R the rectangle of B and D.

Then the rectangles P and R, being between the fame parallels, are to each other as their bafes A and B (in. 79) ; and the rectangles Q and R, being between the fame parallels, are to each other as their bafes C and D. But the ratio of A to B is the fame as the ratio of C to D by hypothefis ; therefore the ratio of P to R, is the fame as the ratio of Q to R ; and confequently the rectangles P and Q are equal.   Q. E. D.

Again, if the rectangle of A and D, be equal to the rectangle of B and C ; thefe lines will be proportional, or A : B :: C : D.

For, the rectangles being placed the fame as before : then, becaufe parallelograms between the fame parallels, are to one another as their bafes, the rectangle P : R :: A : B, and Q : R :: C : D. But as P and Q are equal, by fuppofition, they have the fame ratio to R, that is, the ratio of A to B is equal to the ratio of C to D, or A : B :: C : D. Q. E. D.

*Corol.* 1. When the two means, namely, the fecond and third terms, are equal, their rectangle becomes a fquare of the fecond term, which fupplies the place of both the fecond and third.   And hence it follows, that when three lines are proportionals, the rectangle of the two extremes is equal to
                                                                the

the fquare of the mean; and, converfely, if the rectangle of the extremes be equal to the fquare of the mean, the three lines are proportionals.

*Corol.* 2. Since it appears, by the rules of proportion in Arithmetic and Algebra, that when four quantities are proportional, the product of the extremes is equal to the product of the two means; and, by this theorem, the rectangle of the extremes is equal to the rectangle of the two means; it follows, that the area or fpace of a rectangle is reprefented or expreffed by the product of its length and breadth multiplied together. And, in general, a rectangle in geometry is fimilar to the product of the meafures of its two dimenfions of length and breadth, or bafe and height. Alfo, a fquare is fimilar to, or reprefented by, the meafure of its fide multiplied by itfelf. So that, what is fhewn of fuch products, is to be underftood of the fquares and rectangles.

*Corol.* 3. Since the fame reafoning, as in this theorem, holds for any parallelograms whatever, as well as for the rectangles, the fame property belongs to all kinds of parallelograms, having equal angles, and alfo to triangles, which are the halves of parallelograms; namely, that If the fides about the equal angles of parallelograms, or triangles, be reciprocally proportional, the parallelograms or triangles will be equal; and, converfely, if the parallelograms or triangles be equal, their fides about the equal angles will be reciprocally proportional.

*Corol.* 4. Parallelograms, or triangles, having an angle in each equal, are to each other as the rectangles of the fides which are about thefe equal angles.

### THEOREM LXXXII.

If a Line be drawn in a Triangle Parallel to one of its Sides, it will cut the two other Sides Proportionally.

LET DE be parallel to the fide BC of the triangle ABC; then will AD:DB::AE:EC.

For, draw BE and CD. Then the triangles DBE, DCE are equal to each other, becaufe they have the fame bafe DE, and are between the fame parallels DE, BC (th. 25). But the two triangles ADE, BDE, on the bafes AD, DB, have the fame altitude;
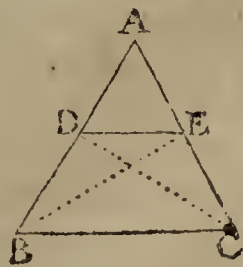
tude ; and the two triangles ADE, CDE, on the bafes AE, EC, have alfo the fame altitude ; and becaufe triangles of the fame altitude are to each other as their bafes, therefore

the triangle ADE : BDE :: AD : DB,
and triangle ADE : CDE :: AE : EC.

But BDE is $=$ CDE ; and equals muft have to equals the fame ratio; therefore AD : DB :: AE : EC.   Q. E. D.

*Corol.* Hence alfo, the whole lines AB, AC, are proportional to their correfponding proportional fegments (corol. th. 66),

namely, AB : AC :: AD : AE,
and        AB : AC :: BD : CE.

## THEOREM LXXXIII.

A RIGHT Line which Bifects any Angle of a Triangle, divides the Side oppofite to the Bifected Angle, into Two Segments, which are Proportional to the two other Adjacent Sides.

Let the angle ACB, of the triangle ABC, be bifected by the line CD, making the angle $r$ equal to the angle $s$ : then will the fegment AD be to the fegment DB, as the fide AC is to the fide CB. Or, - - - AD : DB :: AC : CB.
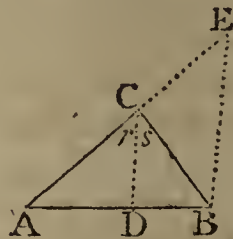
For, let BE be parallel to CD, meeting AC produced at E.  Then, becaufe the line BC cuts the two parallels CD, BE, it makes the angle CBE equal to the alternate angle $s$ (th. 12), and therefore alfo equal to the angle $r$, which is equal to $s$ by the fuppofition.  Again, becaufe the line AE cuts the two parallels DC, BE, it makes the angle E equal to the angle $r$ on the fame fide of it (th. 14). Hence, in the triangle BCE, the angles B and E, being each equal to the angle $r$, are equal to each other, and confequently alfo their oppofite fides CB, CE alfo equal (th. 3).

But now, in the triangle ABE, the line CD, being drawn parallel to the fide BE, cuts the two other fides AB, AE proportionally (th. 82), making AD to DB, as AC to CE or its equal CB.   Q. E. D.

### THEOREM LXXXIV.

Equiangular Triangles are Similar, or have their Like Sides Proportional.

Let ABC, DEF be two equiangular tri-
angles, having the angle A equal to the
angle D, the angle B to the angle E, and
consequently the angle C to the angle F:
then will AB : AC :: DE : DF.

For, make DG = AB, and DH = AC,
and join GH. Then the two triangles
ABC, DGH, having the two sides AB, AC,
equal to the two AG, AH, and the con-
tained angles A and D also equal, are iden-
tical, or equal in all respects (th. 1), namely
the angles B and C are equal to the angles G
and H. But the angles B and C are equal to the angles E
and F by the hypothesis; therefore also the angles G and H
are equal to the angles E and F (ax. 1), and consequently
the line GH is parallel to the side EF (cor. 1, th. 14).

Hence then, in the triangle DEF, the line GH, being
parallel to the side EF, divides the two other sides propor-
tionally, making DG : DH :: DE : DF (cor. th. 82). But
DG and DH are equal to AB and BC; therefore — — —
AB : AC :: DE : DF.    Q. E. D.
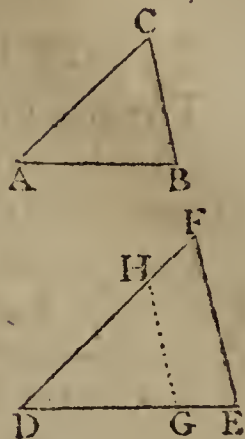
### THEOREM LXXXV.

Triangles which have their Sides Proportional, are Equi-
angular.

In the two triangles ABC, DEF, if
AB : DE :: AC : DF :: BC . EF; the two
triangles will have their corresponding an-
gles equal.

For, if the triangle ABC be not equian-
gular with the triangle DEF, suppose some
other triangle, as DEG, to be equiangular
with ABC. But this is impossible: for if
the two triangles ABC, DEG were equi-
angular, their sides would be proportional
(th. 84). So that, AB being to DE as AC
to DG, and AB to DE as BC to EG, it follows that DG
and EG, being fourth proportionals to the same three quan-
tities,
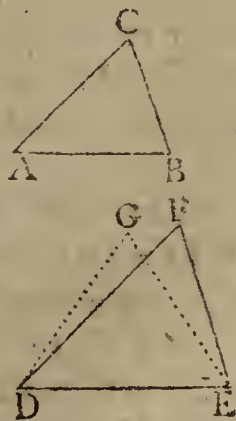
tities, as the two DF, EF, the former DG, EG, would be equal to the latter, DF, EF. Thus then, the two triangles DEF, DEG, having their three fides equal, would be iden-tical (th. 5); which is abfurd, fince their angles are un-equal.

### THEOREM LXXXVI.

TRIANGLES, which have an Angle in the one Equal to an Angle in the other, and the Sides about thefe angles Proportional, are Equiangular.

Let ABC, DEF be two triangles, having the angle A $=$ the angle D, and the fides AB, AC proportional to the fides DE, DF: then will the triangle ABC be equiangular with the triangle DEF.

For, make DG $=$ AB, and DH $=$ AC, and join GH.

Then, the two triangles ABC, DGH, having two fides equal, and the contained angles A and D equal, are identical and equiangular (th. 1), having the angles G and H equal to the angles B and C. But, fince the fides DG, DH are proportional to the fides DE, DF, the line GH is parallel to EF (th. 82); hence the angles E and F are equal to the angles G and H (th. 14), and confequently to their equals B and C. Q. E. D.
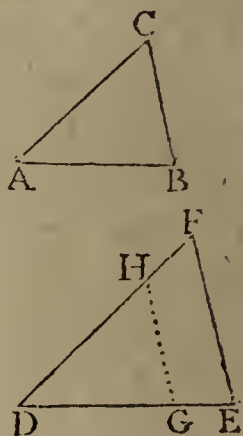
### THEOREM LXXXVII.

IN a Right-Angled Triangle, a Perpendicular from the Right Angle, is a Mean Proportional between the Segments of the Hypothenufe; and each of the Sides, about the Right Angle, is a Mean Proportional between the Adjacent Segment and the Hypothenufe.

Let ABC be a right-angled triangle, and CD a perpendicular from the right angle C to the hypothenufe AB; then will
CD be a mean proportional between AD and DB;
AC a mean proportional between AB and AD;
BC a mean proportional between AB and BD.

Or, AD : CD :: CD : DB; and AB : AC :: AC : AD; and AB : BC :: BC : BD.
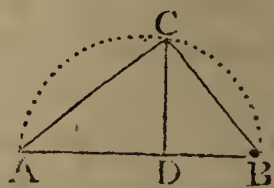
For,

For, the two triangles ABC, ADC, having the right angles at C and D equal, and the angle A common, have their third angles equal, and are equiangular. (cor. 1, th. 17). In like manner, the two triangles ABC, BDC, having the right angles at C and D equal, and the angle B common, have their third angles equal, and are equiangular.

. Hence then, all the three triangles ABC, ADC, BDC, being equiangular, will have their like fides proportional (th. 84);

> viz. AD : CD :: CD : DB;
> and AB : AC :: AC : AD;
> and AB : BC :: BC : BD.     Q. E. D.

*Corol.* Becaufe the angle in a femicircle is a right angle (th. 52); it follows that if, from any point C in the periphery of the femicircle, a perpendicular be drawn to the diameter AB; and the two chords CA, CB, be drawn to the extremities of the diameter: then are AC, BC, CD the mean proportionals as in this theorem, or (by th. 77), $- - -$ $CD^2 = AD . DB$; $AC^2 = AB . AD$; and $BC^2 = AB . BD$.

### THEOREM LXXXVIII.

Equiangular or Similar Triangles, are to each other as the Squares of their Like Sides.

LET ABC, DEF be two equiangular triangles, AB and DE being two homologous or like fides : then will the triangle ABC be to the triangle DEF, as the fquare of AB is to the fquare of DE, or as $AB^2$ to $DE^2$.
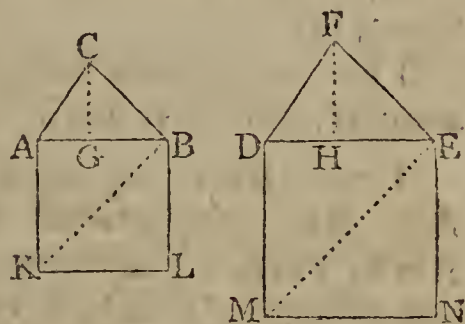
For, let AL and DN be the fquares on AB and DE ; alfo draw their diagonals BK, EM, and the perpendiculars CG, FH of the two triangles.

Then, fince equiangular triangles have their like fides proportional (th. 82), in the two equiangular triangles ABC, DEF, the fide AC : DF :: AB : DE ; and in the two ACG, DFH, the fide AC : DF :: CG : FH ; therefore, by equality, CG : FH :: AB : DE, or CG : AB :: FH : DE.

But becaufe triangles on equal bafes are to each other as their altitudes, the triangles ABC, ABK, on the fame bafe AB, are to each other, as their altitudes CG, AK, or AB;

and the triangles DEF, DEM, on the fame bafe DE, are as their altitudes FH, DM or DE ;

that is, triangle ABC : triangle ABK :: CG : AB,
and triangle DEF : triangle DEM :: FH : DE.

But it has been fhewn that CG : AB :: FH : DE ;
theref. of equality $\triangle$ABC : $\triangle$ABK :: $\triangle$DEF : $\triangle$DEM,
or alternately, as $\triangle$ABC : $\triangle$DEF :: $\triangle$ABK : $\triangle$DEM,

But the fquares AL, DN, being the double of the triangles ABK, DEM, have the fame ratio with them ;

therefore the $\triangle$ABC : $\triangle$ DEF :: fquare AL : fquare DN.

Q. E. D.

## THEOREM LXXXIX.

All Similar Figures are in Proportion to each other, as the Squares of their Like Sides.

LET ABCDE, FGHIK be any two fimilar figures, the like fides being AB, FG, and BC, GH, and fo on in the fame order : then will the figure ABCDE be to the figure FGHIK, as the fquare of AB to the fquare of FG, or as AB² to FG².
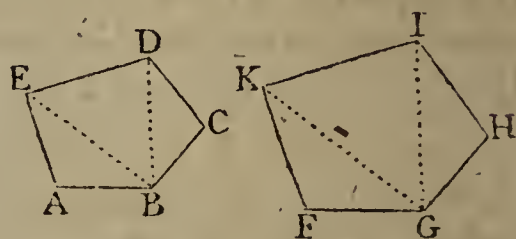
For, draw BE, BD, GK, GI, dividing the figures into an equal number of triangles, by lines from two equal angles B and G.

The two figures being fimilar (by fuppof.), they are equiangular, and have their like fides proportional (def. 67).

Then, fince the angle A is = the angle F, and the fides AB, AE, proportional to the fides FG, FK, the triangles ABE, FGK are equiangular (th. 86). In like manner, the two triangles BCD, GHI, having the angle C = the angle H, and the fides BC, CD proportional to the fides GH, HI, are alfo equiangular. Alfo, if from the equal angles AED, FKI, there be taken the equal angles AEB, FKG, there will remain the equals BED, GKI ; and if from the equal angles CDE, HIK, be taken away the equals CDB, HIG, there will remain the equals BDE, GIK ; fo that the two triangles BDE, GIK, having two angles equal, are alfo equiangular. Hence each triangle of the one figure, is equiangular with each correfponding triangle of the other.

But equiangular triangles are fimilar, and are proportional to the fquares of their like fides (th. 88).

Therefore

Therefore the $\triangle$ ABE : $\triangle$ FGK :: AB$^2$ : FG$^2$,

and $\triangle$ BCD : $\triangle$ GHI :: BC$^2$ : GH$^2$,

and $\triangle$ BDE : $\triangle$ GIK :: DE$^2$ : IK$^2$,

But as the two polygons are similar, their like sides are proportional, and confequently their fquares alfo proportional ; fo that all the ratios AB$^2$ to FG$^2$, and BC$^2$ to GH$^2$, and DE$^2$ to IK$^2$, are equal among themfelves, and confequently the correfponding triangles alfo, ABE to FGK, and BCD to GHI, and BDE to GIK, have all the fame ratio, viz. that of AB$^2$ to FG$^2$ : and hence all the antecedents, or the figure ABCDE, have to all the confequents, or the figure FGHIK, ftill the fame ratio, viz. that of AB$^2$ to FG$^2$ (th. 72).

Q. E. D.

### THEOREM XC.

SIMILAR Polygons Infcribed in Circles, have their Like Sides, and alfo their Whole Perimeters, in the Same Ratio as the Diameters of the Circles in which they are infcribed.

Let ABCDE, FGHIK be two fimilar figures, in-fcribed in the circles, whofe diameters are AL and FM; then will each fide AB, BC, &c, of the one figure, be to the like fide FG, GH, &c, of the
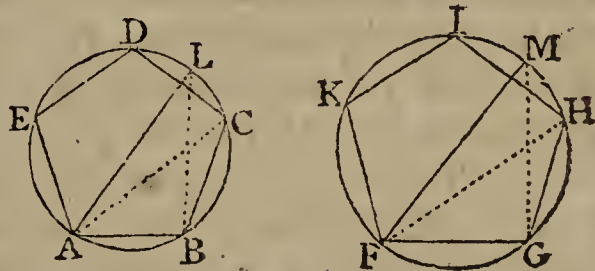


other figure, or the whole perimeter AB + BC, &c, of the one figure, to the whole perimeter FG + GH, &c, of the other figure, as the diameter AL to the diameter FM.

For, draw the two correfponding diagonals AC, FH, as alfo the lines BL, GM. Then, fince the polygons are fimilar, they are equiangular, and their like fides have the fame ratio (def. 67); therefore the two triangles ABC, FGH have the angle B = the angle G, and the fides AB, BC proportional to the two fides FG, GH, and confequently thefe two triangles are equiangular (th. 86), and have the angle ACB = FHG. But the angle ACB = ALB, ftand-ing on the fame arc AB; and the angle FHG = FMG, ftanding on the fame arc FG ; therefore the angle ALB = FMG (ax. 1). And fince the angle ABL = FGM, being both right angles, becaufe in a femicircle ; therefore the two triangles ABL, FGM, having two angles equal, are equi-angular ; and confequently their like fides are proportional

Z 2 (th. 84);

(th. 84); hence AB : FG :: the diameter AL : the diameter
FM.

In like manner, each fide BC, CD, &c. has to each fide
GH, HI, &c, the fame ratio of AL to FM ; and confe-
quently the fums of them are ftill in the fame ratio ; viz.
AB + BC + CD, &c : FG + GH + HI, &c :: the diam.
AL : the diam. FM (th. 72).    Q. E. D.


### THEOREM XCI.

SIMILAR Polygons Infcribed in Circles, are in Proportion
to each other as the Squares of the Diameters of thofe
Circles.

Let ABCDE, FGHIK
be two fimilar figures, in-
fcribed in the circles,
whofe diameters are AL
and FM ; then the furface
of the polygon ABCDE
will be to the furface of
the polygon FGHIK, as $AL^2$ to $FM^2$.
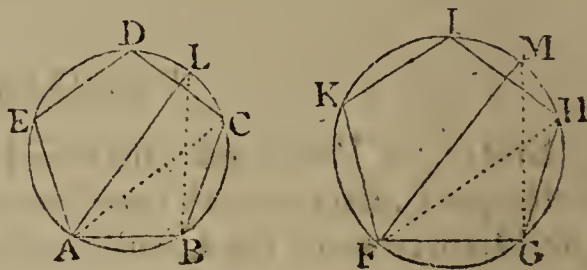
For, the figures, being fimilar, are to each other as the
fquares of their like fides, $AB^2$ to $FG^2$ (th. 88). But, by
the laft theorem, the fides AB, FG, are as the diameters
AL, FM ; and therefore the fquares of the fides $AB^2$ to
$FG^2$, as the fquares of the diameters $AL^2$ to $FM^2$ (th. 74).
Confequently the polygons ABCDE, FGHIK, are alfo to
each other as the fquares of the diameters $AL^2$ to $FM^2$
(ax. 1).    Q. E. D.


### THEOREM XCII.

The Circumferences of Circles are in Proportion to each
other as their Diameters.

LET D, d denote the diameters of two circles, and C, c
their circumferences ;

then will D : d :: C : c, or D : C :: d : c.

For, by theor. 90, fimilar polygons infcribed in circles,
have their perimeters in the ratio of the diameters of thofe
circles.

Now, as this property belongs to all polygons, whatever
the number of the fides may be ; conceive the number of the
fides to be indefinitely great, and the length of each inde-
finitely fmall, till they coincide with the circumference of
                                                        the

the circle, and be equal to it, indefinitely near. Then the perimeter of the polygon of an infinite number of fides, is the fame thing as the circumference of the circle. Hence it appears that the circumferences of the circles, being the fame as the perimeters of fuch polygons, are as the diameters of the circles. Q. E. D.

THEOREM XCIII.

The Areas or Spaces of Circles, are to each other as the Squares of their Diameters, or of their Radii.

LET A, a be the areas or fpaces of two circles, and D, d their diameters; then $A : a :: D^2 : d^2$.

For, by theorem 91, fimilar polygons infcribed in circles are to each other as the fquares of the diameters of the circles.

Hence, conceiving the number of the fides of the polygons to be increafed more and more, or the length of the fides to become lefs and lefs, the polygon approaches nearer and nearer to the circle, till at length, by an infinite approach, they coincide and become in effect equal; and then it follows, that the fpaces of the circles, which are the fame as of the polygons, will be to each other as the fquares of the diameters of the circles. Q. E. D.

*Corol.* The fpaces of circles are alfo to each other as the fquares of the circumferences; fince the circumferences are in the fame proportion as the diameters, by theorem 92.

THEOREM XCIV.

Every Circle is Equal to the Rectangle of its Radius and a Right Line equal to Half its Circumference.

CONCEIVE a regular polygon to be infcribed in the circle; and radii drawn to all the angular points, dividing it into as many equal triangles as the polygon has fides, one of which is ABC, of which the altitude is the perpendicular CD from the centre to the bafe AB.

Then the triangle ABC, being equal to half a rectangle of equal bafe and altitude (th. 26), is equal to half the rectangle of the bafe AB and altitude CD, or

æqual

equal to the rectangle of the altitude CD and half the bafe AB; confequently the whole polygon, or all the triangles added together which compofe it, is equal to the rectangle of the common altitude CD, and the halves of all the fides, or the half of the perimeter of the polygon.

Now, conceive the number of fides of the polygon to be indefinitely increafed; then will its perimeter coincide with the circumference of the circle, and confequently the altitude AD will become equal to the radius, and the whole polygon equal to the circle. Confequently the fpace of the circle, or of the polygon in that ftate, is equal to the rectangle of the radius and half the circumference. Q. E. D.

## OF PLANES AND SOLIDS.

### DEFINITIONS.

DEF. 89. THE Common Section of two Planes, is the line in which they meet, to cut each other.

90. A Right Line is Perpendicular to a Plane, when it is perpendicular to every right line which meets it in that plane.

91. One Plane is Perpendicular to Another, when every right line in the one, which is perpendicular to the line of their common fection, is perpendicular to the other.

92. The Inclination of one Plane to another, or the angle they form between them, is the angle contained by two right lines, drawn from any point in the common fection, and at right angles to the fame, one of thefe lines in each plane.

93. Parallel Planes, are fuch as being produced ever fo far both ways will never meet, or which are every where at an equal perpendicular diftance.

94. A Solid Angle, is that which is made by three or more plane angles, meeting each other in the fame point.

95. Si-

95. Similar Solids, contained by plane figures, are such as have all their folid angles equal, each to each, and are bounded by the fame number of fimilar planes.

96. A Prifm, is a folid whofe ends are parallel, equal, and like plane figures, and its fides, connecting thofe ends, are parallelograms.

97. A Prifm takes particular names according to the figure of its bafe or ends, whether triangular, fquare, rectangular, pentagonal, hexagonal, &c.

98. An Upright Prifm, is that which has the planes of the fides perpendicular to the planes of the ends or bafe.

99. A Parallelopiped, or Parallelopipedon, is a prifm bounded by fix parallelograms, every oppofite two of which are equal, alike, and parallel.

100. A Rectangular Parallelopipedon, is that whofe bounding planes are all rectangles, which are perpendicular to each other.

101. A Cube, is a fquare prifm, being bounded by fix equal fquare fides or faces, and are perpendicular to each other.
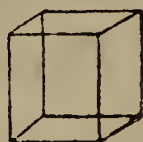
102. A Cylinder is a round prifm, having circles for its ends; and is conceived to be formed by the revolution of a right line about the circumferences of two equal and parallel circles, always parallel to the axis.

103. The Axis of a Cylinder, is the right line joining the centres of the two parallel circles, about which the figure is defcribed.

104. A Pyramid, is a folid, whofe bafe is any right-lined plane figure, and its fides triangles, having all their vertices meeting together in a point above the bafe, called the Vertex of the pyramid.

105. A pyramid, like the prifm, takes particular names from the figure of the bafe.

106. A Cone, is a round pyramid, having a circular bafe; and is conceived to be generated by the revolution of a right line about the circumference of a circle, one end of which is fixed at a point above the plane of that circle.

107. The

107. The Axis of a cone, is the right line joining the vertex, or fixed point, and the centre of the circle about which the figure is defcribed.

108. Similar Cones and Cylinders, are fuch as have their altitudes and the diameters of their bafes proportional.

109. A Sphere, is a folid bounded by one convex furface, which is every where equally diftant from a certain point within, called the Centre. It is conceived to be generated by the revolution of a femicircle about its diameter, which remains fixed.

110. The Axis of a Sphere, is the right line about which the femicircle revolves; and the centre is the fame as that of the revolving femicircle.

111. The Diameter of a Sphere, is any right line paffing through the centre, and terminated both ways by the furface.

112. The Altitude of a Solid, is the perpendicular drawn from the vertex to the oppofite fide or bafe.

## THEOREM XCV.

A Perpendicular is the Shorteft Line which can be drawn from any Point to a Plane.

LET AB be perpendicular to the plane DE; then any other line as AC, drawn from the fame point A to the plane, will be longer than the line AB.



In the plane draw the line BC, joining the points B, C.

Then, becaufe the line AB is perpendicular to the plane DE, the angle B is a right angle (def. 90); and therefore the line AB is lefs than any other line AC (th. 21). Q. E. D.
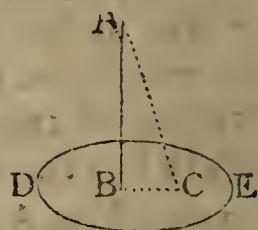
## THEOREM XCVI.

A Perpendicular Meafures the Diftance of any Point from a Plane.

THE diftance of one point from another is meafured by a right line joining them, becaufe this is the fhorteft line which can be drawn from one point to another. So, alfo, the diftance from a point to a line, is meafured by a perpendicular, becaufe this line is the fhorteft which can be drawn

from

from the point to the line. In like manner, the diftance from a point to a plane, muft be meafured by a perpendicular drawn from that point to the plane, becaufe this is the fhorteft line which can be drawn from the point to the plane.

### THEOREM XCVII.

The Common Section of Two Planes, is a Right Line.

LET ACBDA, AEBFA be two planes cutting each other, and A, B two points in which the two planes meet; drawing the line AB, this line will be the common interfection of the two planes.
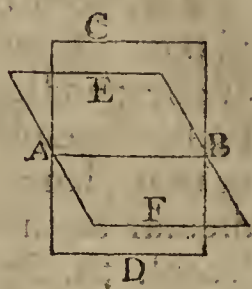
For, becaufe the right line AB touches the two planes in the points A and B, it touches them in all other points (def. 20): this line is therefore common to the two planes. That is, the common interfection of the two planes is a right line.

Q. E. D.

### THEOREM XCVIII.

IF a Right Line be Perpendicular to two other Right Lines, at their Common Point of Meeting; it will be Perpendicular to the Plane of thofe Right Lines.

Let the line AB make right angles with the lines AC, AD; it will be perpendicular to the plane CDE which paffes through thefe lines.

If the line AB were not perpendicular to the plane CDE, another plane might pafs through the point A, to which the line AB would be perpendicular. But this is impoffible; for, fince the angles BAC, BAD are right angles, this other plane muft pafs through the points C, D. Hence, this plane paffing through the two points A, C of the line AC, and through the two points A, D of the line AD, it will pafs through both thefe two lines, and therefore be the fame plane with the former. Q. E. D.
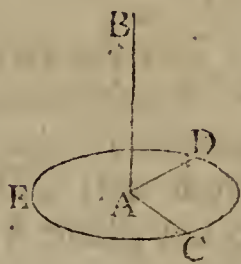
## THEOREM XCIX.

If Two Right Lines be Perpendicular to the Same Plane, they will be Parallel to each other.

LET the two lines AB, CD be both of them perpendicular to the same plane EBDF; then will AB be parallel to CD.
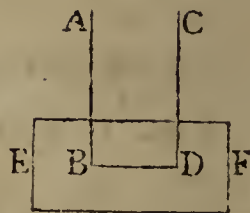
For, join B, D, by the line BD in the plane. Then, because the lines AB, CD are perpendicular to the plane EF, they are both perpendicular to the line BD (def. 90) in that plane. And because AB and CD are both perpendicular to the same line BD, they are parallel to each other (corol. th. 13). Q. E. D.

*Corol.* If two lines be parallel, and if one of them be perpendicular to any plane, the other will also be perpendicular to the same plane.

## THEOREM C.

IF Two Planes Cut each other at Right Angles, and a Right Line be drawn in one of the Planes Perpendicular to their Common Intersection, it will be Perpendicular to the other Plane.

Let the two planes ACBD, AEBF cut each other at right angles; and the line CG perpendicular to their common section AB; then will CG be also perpendicular to the other plane AEBF.
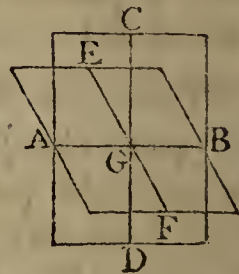
For, draw EG perpendicular to AB. Then, because the two lines GC, GE are perpendicular to the common intersection AB, the angle CGE is the angle of inclination of the two planes (def. 92). But since the two planes cut each other perpendicularly, the angle of inclination CGE is a right angle. And since the line CG is perpendicular to the two lines GA, GE, in the plane AEBF, it is therefore perpendicular to that plane (th. 98). Q. E. D.

THE-

## THEOREM CI.

IF one Plane Meet another Plane; it will make Angles with that other Plane, which are together Equal to Two Right Angles.

Let the plane ACB meet the plane AEBF; thefe planes make with each other two angles whofe fum is equal to two right angles.

For, through any point G, in the common fe&tion AB, draw CD, EF perpendicular to AB. Then, the line CG makes with EF two angles together equal to two right angles. But thefe two angles are (by def. 92), the angles of inclination of the two planes. Therefore the two planes make angles with each other, which are together equal to two right angles.

*Corol.* In like manner, it may be demonftrated, that planes which interfe&t, have their vertical or oppofite angles equal; alfo, that parallel planes have their alternate angles equal; and fo on, as in parallel lines.

## THEOREM CII.

IF Two Planes be Parallel to each other; a Right Line which is Perpendicular to one of the Planes, will alfo be Perpendicular to the other.

Let the two planes CD, EF be parallel, and let the line AB be perpendicular to the plane CD; then fhall it alfo be perpendicular to the other plane EF.



For, from any point G, in the plane EF, draw GH perpendicular to the plane CD, and draw AH, BG.
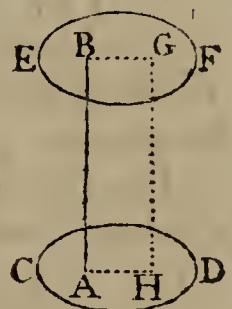
Then, becaufe BA, GH are both perpendicular to the plane CD, the angles A and H are both right angles. And becaufe the planes CD, EF are parallel, the perpendiculars BA, GH are equal (def. 93). Hence it follows that the lines BG, AH are parallel (def. 9). And the line AB being perpendicular to the line AH, is alfo perpendicular to the parallel line BG (cor. th. 12).

In like manner, it is proved, that the line AB is perpendicular to all other lines which can be drawn from the point

B

B in the plane EF. Therefore the line AB is perpendicular
to the plane EF (def. 90).   Q. E. D.

### THEOREM CIII.

If Two Right Lines be Parallel to a Third Line, though
not in the fame Plane with it ; they will be Parallel to each
other.

Let the right lines AB, CD be each of
them parallel to the third line EF, though
not in the fame plane with it ; then will AB
be parallel to CD.

For, from any point G in the line EF, let
GH, GI be each perpendicular to EF, in the
planes AF, ED of the propoſed parallels.

Then, ſince the line EF is perpendicular
to the two lines GH, GI, it is perpendicular
to the plane GHI of thoſe lines (th. 98). And becauſe EF
is perpendicular to the plane GHI, its parallel AB is alſo
perpendicular to that plane (th. 99, cor.) For the ſame
reaſon, the line CD is perpendicular to the ſame plane GHI.
Hence, becauſe the two lines AB, CD are perpendicular to
the ſame plane, theſe two lines are parallel (th. 99).

                                Q. E. D.

### THEOREM CIV.

If Two Lines that meet each other, be Parallel to Two
other Lines that meet each other, though not in the ſame
Plane with them ; the Angles contained by thoſe Lines will
be equal.

Let the two lines AB, BC be parallel to
the two lines DE, EF ; then will the angle
ABC be equal to the angle DEF.

For, make the lines AB, BC, DE, EF all
equal to each other ; and join AC, DF, AD,
BE, CF.
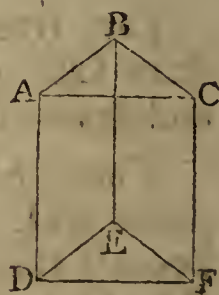
Then, the lines AD, BE, joining the equal
and parallel lines AB, DE, are equal and
parallel (th. 24). For the ſame reaſon, CF, BE are equal
and parallel. Therefore AD, CF are equal and parallel
(th. 15) ; and conſequently alſo AC, DF (th. 24). Hence,
the two triangles ABC, DEF, having all their ſides equal,
                                       each

each to each, have their angles alfo equal, and confequently the angle ABC == the angle DEF.   Q. E. D.

## THEOREM CV.

The Sections made by a Plane cutting two other Parallel Planes, are alfo Parallel to each other.

LET the two parallel planes AB, CD be cut by the third plane EFHG, in the lines EF, GH; thefe two fections EF, GH will be parallel.
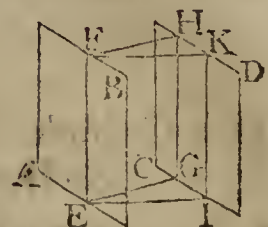
Suppofe EG, FH be drawn parallel to each other in the plane EFHG; alfo let EI, FK be perpendicular to the plane CD; and let IG, KH be joined.

Then EG, FH being parallels, and EI, FK, being both perpendicular to the plane CD, are alfo parallel to each other (th. 99); confequently the angle HFK is equal to the angle GEI (th. 104). But the angle FKH is alfo equal the angle EIG, being both right angles; therefore the two triangles are equiangular (cor. 1, th. 17); and the fides FK, EI being the equal diftances between the parallel planes (def. 93), it follows that the fides FH, EG are alfo equal (th. 2). But thefe two lines are parallel (by fuppof.), as well as equal; and confequently the two EF, GH, joining thofe equal parallels, are alfo parallel (th. 24).   Q. E. D.

## THEOREM CVI.

If a Prifm be cut by a Plane Parallel to its Bafe, the Section will be Equal and Like to the Bafe.

LET AG be any prifm, and IL a plane parallel to the bafe AC; then will the plane IL be equal and like to the bafe AC, or the two planes will have all their fides and all their angles equal.
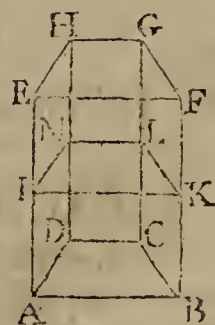
For, the two planes AC, IL being parallel, by hypothefis; and two parallel planes, cut by a third plane, having parallel fections (th. 105); therefore IK is parallel to AB, KL to BC, LM to CD, and IM to AD. But AI and BK are parallels (by def. 96); confequently AK is a parallelogram; and the oppofite fides AB, IK are equal (th. 22).

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In like manner, it is ſhewn that KL is $=$ BC and LM $=$ CD,
and IM $=$ AD, or the two planes AC, IL are mutually equi-
lateral. But theſe two planes, having their correſponding
ſides parallel, have the angles contained by them alſo equal
(th. 104), namely, the angle A $=$ the angle I, the angle
B $=$ the angle K, the angle C $=$ the angle L, and the angle
D $=$ the angle M. So that the two planes AC, IL have all
their correſponding ſides and angles equal, or are equal and
like. Q. E. D.

## THEOREM CVII.

If a Cylinder be cut by a Plane Parallel to its Baſe, the
Section will be a Circle, Equal to the Baſe.

Let AF be a cylinder, and GHI any
ſection parallel to the baſe ABC; then will
GHI be a circle, equal to ABC.

For, let the planes KE, KF paſs through
the axis of the cylinder MK, and meet the
ſection GHI in the three points H, I, L;
and join the points as in the figure.
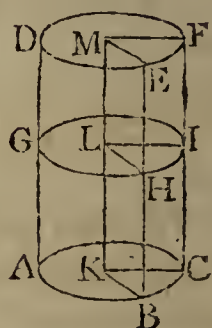
Then, ſince KL, CI are parallel (by
def. 102): and the plane KI, meeting the
two parallel planes ABC, GHI, makes the two ſections KC,
LI parallel (th. 105); the figure KLIC is therefore a pa-
rallelogram, and conſequently has the oppoſite ſides LI, KC
equal, where KC is a radius of the circular baſe.

In like manner, it is ſhewn that LH is equal to the radius
KB; and that any other lines, drawn from the point L to
the circumference of the ſection GHI, are all equal to radii
of the baſe; conſequently GHI is a circle, and equal to
ABC. Q. E. D.

## THEOREM CVIII.

All Priſms and Cylinders, of Equal Baſes and Altitudes, are
Equal to each other.

Let AC, DF be two
priſms, and a cylinder,
upon equal baſes AB,
DE, and having equal
altitudes; then will the
ſolids AC, DF be equal.

For, let PQ, RS be
any
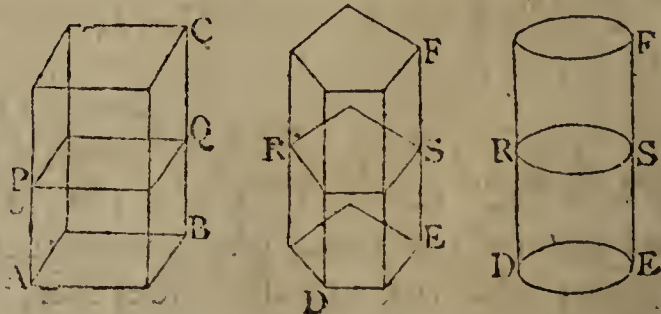
any two sections parallel to the bases, and equidistant from them. Then, by the last two theorems, the section PQ is equal to the base AB, and the section RS equal the base DE. But the bases AB, DE are equal, by the hypothesis; therefore the sections PQ, RS are equal also. And in like manner, it may be shewn, that any other corresponding sections are equal to one another.

Since then every section in the prism AC, is equal to its corresponding section in the prism, or cylinder RS, the prisms and cylinder themselves, which are composed of those sections, must also be equal. Q. E. D.

*Corol.* Every prism, or cylinder, is equal to a rectangular parallelopipedon, of an equal base and altitude.

### THEOREM CIX.

Rectangular Parallelopipedons, of Equal Altitudes, have to each other the same Proportion as their Bases.

LET AC, EG be two rectangular parallelopipedons, having the equal altitudes AD, EH; then will AC be to EG as the base AB is to the base EF.
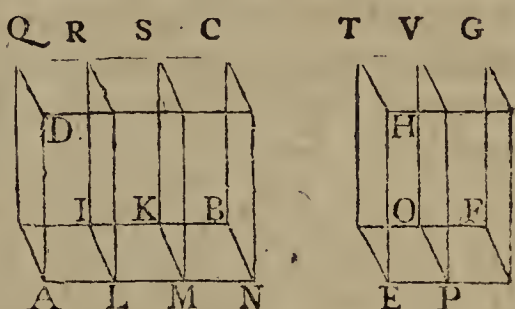


For, let the proportion of the base AB to the base EF, be that of any one number $m$ (3) to any other number $n$ (2). And conceive AB to be divided into $m$ equal parts, or rectangles, AI, LK, MB (by dividing AN into that number of equal parts, and drawing IL, KM parallel to BN). And let EF be divided, in like manner, into $n$ equal parts, or rectangles, EO, PF: All of these parts of both bases being mutually equal among themselves. And through the lines of division let the plane sections LR, MS, PV pass parallel to AQ, ET.

Then the parallelopipedons AR, LS, MC, EV, PG are all equal, having equal bases and heights. Therefore the solid AC is to the solid EG, as the number of parts in AC to the number of equal parts in EG, or as the number of parts in AB to the number of equal parts in EF, that is, as the base AB to the base EF. Q. E. D.

*Corol.* From this theorem, and the corollary to the last, it appears, that all prisms and cylinders of equal altitudes, are

to each other as their bafes ; every prifm and cylinder being equal to a rectangular parallelopipedon of an equal bafe and height.

### THEOREM CX.

Rectangular Parallelopipedons, of Equal Bafes, are in Proportion to each other as their Altitudes.

LET AB, CD be two rectangular parallelopipedons ftanding on the equal bafes AE, CF ; then will AB be to CD as the altitude EB is to the altitude DF.
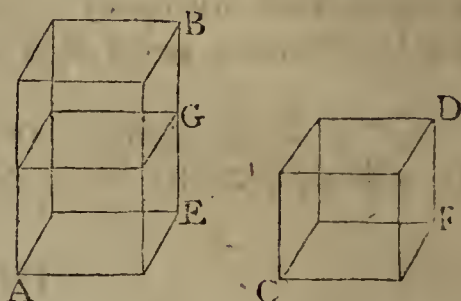


For, let AG be a rectangular parallelopipedon on the bafe AE, and its altitude EG equal to the altitude FD of the folid CD.

Then AG and CD are equal, being prifms of equal bafes and altitudes. But if HB, HG be confidered as bafes, the folids AB, AG, of equal altitude AH, will be to each other as thofe bafes HB, HG. But thefe bafes HB, HG, being parallelograms of equal altitude HE, are to each other as their bafes EB, EG ; and therefore the two prifms AB, AG are to each other as the lines EB, EG. But AG is equal CD, and EG equal FD ; confequently the prifms AB, CD are to each other as their altitudes EB, FD ; that is, — — AB : CD :: EB : FD. Q. E. D.

*Corol.* 1. From this theorem, and the corollary to theorem 108, it appears, that all prifms and cylinders, of equal bafes, are to one another as their altitudes.

*Corol.* 2. Becaufe, by corollary 1, prifms and cylinders are as their altitudes, when their bafes are equal. And, by the corollary to the laft theorem, they are as their bafes, when their altitudes are equal. Therefore, univerfally, when neither are equal, they are to one another as the product of their bafes and altitudes. And hence alfo thefe products are the proper numeral meafures of their quantities or magnitudes.

### THEOREM CXI.

SIMILAR Prifms and Cylinders are to each other as the Cubes of their Altitudes, or of any other Like Linear Dimenfions.

L t

Let ABCD, EFGH be two fimilar prifms; then will the prifm CD be to the prifm GH, as $AB^3$ to $EF^3$, or as $AD^3$ to $EH^3$.
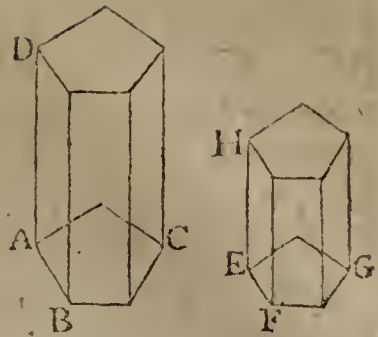
For the folids are to each other as the product of their bafes and altitudes (th. 110, cor. 2), that is, as $AC . AD$ to $EG . EH$. But the bafes, being fimilar planes, are to each other as the fquares of their like fides, that is, $AC$ to $EG$ as $AB^2$ to $EF^2$; therefore the folid $CD$ is to the folid $GH$ as $AB^2 . AD$ to $EF^2 . EH$. But BD and FH, being fimilar planes, have their like fides proportional, that is, $AB : EF :: AD : EH$, or $AB^2 : EF^2 :: AD^2 : EH^2$: therefore — — — — — $AB^2 . AD : EF^2 . EH :: AB^3 : EF^3$, or $:: AD^3 : EH^3$; and confequently the folid $CD$ : folid $GH :: AB^3 : EF^3 :: AD^3 : EH^3$.

Q. E. D.

## THEOREM CXII.

In a Pyramid, a Section Parallel to the Bafe is fimilar to the Bafe; and thefe two planes are to each other as the Squares of their Diftances from the Vertex.

Let ABCD be a pyramid, and EFG a fection parallel to the bafe BCD, alfo AIH a line perpendicular to the two planes at H and I; then will BD, EG be two fimilar planes, and the plane BD will be to the plane EG as $AH^2$ to $AI^2$.
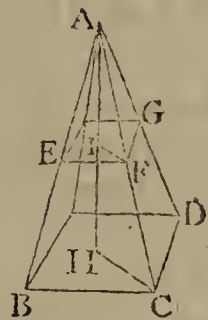
For, join CH, FI. Then, becaufe a plane cutting two parallel planes, makes parallel fections (th. 105), therefore the plane ABC, meeting the two parallel planes BD, EG, makes the fections BC, EF parallel : In like manner, the plane ACD makes the fections CD, FG parallel. Again, becaufe two pair of parallel lines make equal angles (th. 104), the two EF, FG, which are parallel to BC, CD, make the angle EFG equal the angle BCD. And, in like manner, it is fhewn, that each angle in the plane EG is equal to each angle in the plane BD, and confequently thofe two planes are equiangular.

Again, the three lines AB, AC, AD, making with the parallels BC, EF, and CD, FG, equal angles (th. 14), and the angles at A being common, the two triangles ABC, AEF

are equiangular, as alfo the two triangles ACD, AFG, and have therefore their like fides proportional, namely, — — AC : AF :: BC : EF :: CD : FG. And, in like manner, it may be fhewn, that all the lines in the plane EG are proportional to all the correfponding ones in the bafe BD. Hence thefe two planes, having their angles equal and their fides proportional, are fimilar by def. 68.

But, fimilar planes being to each other as the fquares of their like fides, the plane $BD : EG :: BC^2 : EF^2$ or $:: AC^2 : AF^2$, by what is fhewn above. Alfo, the two triangles AHC, AIF, having the angles H and I right ones (th. 98), and the angle A common, are equiangular, and have therefore their like fides proportional, namely, AC:AF::AH:AI, or $AC^2 : AF^2 :: AH^2 : AI^2$. Confequently the two planes BD, EG, which are as the former fquares $AC^2$, $AF^2$, will be alfo as the latter fquares $AH^2$, $AI^2$, that is, — — — $BD : EG :: AH^2 : AI^2$.　　　　　　　　　　　　　Q. E. D.

### THEOREM CXIII.

In a Cone, any Section Parallel to the Bafe is a Circle; and this Section is to the Bafe as the Squares of their Diftances from the Vertex.

Let ABCD be a cone, and GHI a fection parallel to the bafe BCD; then will GHI be a circle, and BCD, GHI will be to each other as the fquares of their diftances from the vertex.

For, draw ALF perpendicular to the two parallel planes; and let the planes ACE, ADE pafs through the axis of the cone AKE, meeting the fection in the three points H, I, K.

Then, fince the fection GHI is parallel to the bafe BCD, and the planes CK, DK meet them, HK is parallel to CE, and IK to DE (th. 105). And becaufe the triangles formed by thefe lines are equiangular, KH:EC::AK:AE::KI:ED. But EC is equal to ED, being radii of the fame circle; therefore KI is alfo equal to KH. And the fame may be fhewn of any other lines drawn from the point K to the circumference of the fection GHI, which is therefore a circle.

Again, by fimilar triangles, AL : AF :: AK : AE or :: KI:ED, hence $AL^2:AF^2::KI^2:ED^2$; but $KI^2:ED^2::$
　　　　　　　　　　　　　　　　　　　　　　　circle

circle GHI : circle BCD (th. 93); therefore AL² : AF² :: circle GHI : circle BCD.   Q. E. D.

## THEOREM CXIV.

All Pyramids, and Cones, of Equal Bafes and Altitudes, are Equal to one another.

LET ABC, DEF be any pyramids and cone, of equal bafes BC, EF, and equal altitudes AG, DH: then will the pyramids and cone ABC and DEF, be equal.
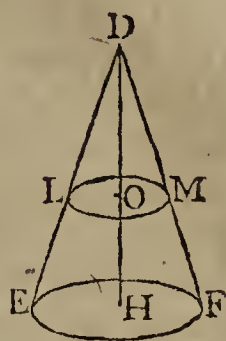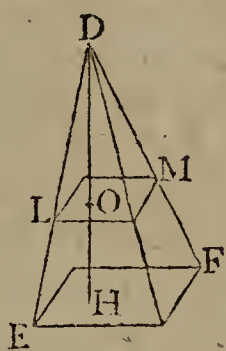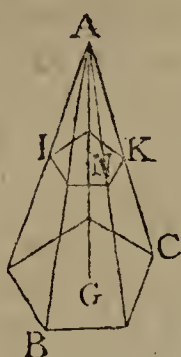
For, parallel to the bafes, and at equal diftances AN, DO, from the vertices, fuppofe the planes IK, LM to be drawn.

Then, by the two preceding theorems, - - - - -
DO² DH² :: LM : EF, and
AN² AG² :: IK : BC.
But fince AN², AG² are equal to DO², DH², .
therefore IK : BC :: LM : EF.  But BC is equal to EF, by hypothefis ; therefore IK is alfo equal to LM.

In the fame manner it is fhewn, that any other fections, at equal diftance from the vertex, are equal to each other.

Since then, every fection in the cone, is equal to the corresponding fection in the pyramids, and the heights are equal, the folids ABC, DEF, compofed of thofe fections, muft be equal alfo.   Q. E. D.

## THEOREM CXV.

Every Pyramid of a Triangular Bafe, is the Third Part of a Prifm of the Same Bafe and Altitude.

LET ABCDEF be a prifm, and BDEF a pyramid, on the fame triangular bafe DEF : then will the pyramid BDEF be a third part of the prifm ABCDEF.

For, in the planes of the three fides of the prifm, draw the diagonals BF, BD, CD. Then the two planes BDF, BCD divide the whole prifm into the three pyramids BDEF,

A a 2                          DABC,
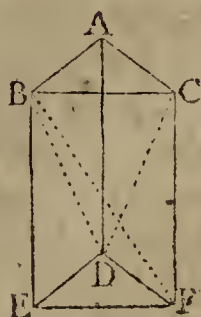
DABC, DBCF, which are proved to be all equal to one another, as follows.

Since the oppofite ends of the prifm are equal to each other, the pyramid whofe bafe is ABC and vertex D, is equal to the pyramid whofe bafe is DEF and vertex B (th. 11_c), being pyramids of equal bafe and altitude.

But the latter pyramid, whofe bafe is DEF and vertex B, is the fame folid as the pyramid whofe bafe is BEF and vertex D, and this is equal to the third pyramid whofe bafe is BCF and vertex D, being pyramids of the fame altitude and equal bafes BEF, BCF.

Confequently all the three pyramids, which compofe the prifm, are equal to each other, and each pyramid is the third part of the prifm, or the prifm is triple of the pyramid.   Q. E. D.

*Corol.* 1. Every pyramid, whatever its figure may be, is the third part of a prifm of the fame bafe and altitude; fince the bafe of the prifm, whatever be its figure, may be divided into triangles, and the whole folid into triangular prifms and pyramids.

*Corol.* 2. Any cone is the third part of a cylinder, or of a prifm, of equal bafe and altitude; fince it has been proved that a cylinder is equal to a prifm, and a cone equal to a pyramid, of equal bafe and altitude.

*Scholium.* Whatever has been demonftrated of the proportionality of prifms, or cylinders, holds equally true of pyramids, or cones ; the former being always triple the latter ; viz. that fimilar pyramids or cones, are as the cubes of their like linear fides, or diameters, or altitudes, &c.

## THEOREM CXVI.

If a Sphere be cut by a Plane, the Section will be a Circle.

LET the fphere AEBF be cut by the plane ADB ; then will the fection ADB be a circle.

Draw the chord AB, or diameter of the fection ; perpendicular to which, or to the fection ADB, draw the axis of the fphere ECGF, through the centre C, which will bifect the chord AB in the point G (th. 41). Alfo, join CA, CB ;

and

and draw CD, GD to any point D in the circumference of the fection ADB.

Then, becaufe CG is perpendicular to the plane ADB, it is perpendicular both to GA and GD (def. 90). So that CGA, CGD are two right-angled triangles, having the perpendicular CG common, and the two hypothenufes CA, CD equal, being both radii of the fphere; therefore the third fides GA, GD are alfo equal (cor. th. 45). In like manner, it is fhewn, that any other line drawn from the centre G to the circumference of the fection ADB, is equal to GA or GB; confequently that fection is a circle.

*Corol.* 1. The centre of every fection of a fphere is always in a diameter of the fphere.

*Corol.* 2. The fection through the centre, is a circle having the fame centre and diameter as the fphere, and is called a great circle of the fphere.

### THEOREM CXVII.

Every Sphere is Two-Thirds of its Circumfcribing Cylinder.

LET ABCD be a cylinder, circum-fcribing the fphere EFGH; then will the fphere EFGH be two-thirds of the cylinder ABCD.

For, let the plane AC be a fection of the fphere and cylinder through the centre I. Join AI, BI. Alfo, let FIH be parallel to AD or BC, and EIG and KL parallel to AB or DC the bafe of the cylinder, the latter line KL meeting BI in M, and the circular fection of the fphere in N.

Then, if the whole plane HFBC be conceived to revolve about the line HF as an axis, the fquare FG will defcribe a cylinder AG, the quadrant IFG will defcribe a hemifphere EFG, and the triangle IFB will defcribe a cone IAB. Alfo, in the revolution, the three lines or parts KL, KN, KM, as radii, will defcribe correfponding circular fections of thofe folids, namely, KL a fection of the cylinder, KN a fection of the fphere, and KM a fection of the cone.

Now, FB being equal to FI or IG, and KL parallel to FB, by fimilar triangles, IK is equal to KM (th. 82). And fince, in the right-angled triangle IKN, IN² is equal IK² + KN² (th. 34); and becaufe KL is equal the radius IG or

IN,

IN, and KM $=$ IK, therefore KL$^2$ is equal KM$^2$ $+$ KN$^2$, or the fquare of the longeft radius, of the faid circular fections, is equal to the fum of the fquares of the two others. And becaufe circles are to each other as the fquares of their diameters, or of their radii, therefore the circle defcribed by KL, is equal to both the circles defcribed by KM and KN, or the fection of the cylinder equal to the correfponding fections of the fphere and cone. And as this is always the cafe in every parallel pofition of KL, it follows, that the cylinder EB, which is compofed of all the former fections, is equal to the hemifphere EFG and cone IAB, which are compofed of all the latter fections.

But the cone IAB is a third part of the cylinder EB (cor. 2, th. 115); confequently the hemifphere EFG is equal to the remaining two-thirds; or the whole fphere EFGH equal to two-thirds of the whole cylinder ABCD. Q. E. D.

*Corol.* 1. A cone, hemifphere, and cylinder, of the fame bafe and altitude, are to each other as the numbers 1, 2, 3.

*Corol.* 2. All fpheres are to each other as the cubes of their diameters; thefe being like all parts of their circumfcribing cylinders.

*Corol.* 3. From the foregoing demonftration it alfo appears, that the fpherical zone or fruftum, EGNP, is equal to the difference between the cylinder EGLO and the cone EMQ, all of the fame common height IK. And that the fpherical fegment PFN, is equal to the difference between the cylinder ABLO and the conic fruftum AQMB, all of the fame common altitude FK.

# PROBLEMS.

## PROBLEM I.

To Bisect a Given Line AB; that is, to divide it into two Equal Parts.

FROM the two centres A and B, with any equal radii, defcribe arcs of circles, interfecting each other in C and D; and draw the line CD, which will bifect the given line AB in the point E.
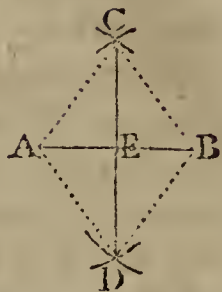
For, draw the radii AC, BC, AD, BD. Then, becaufe all thefe four radii are equal, and the fide CD common, the two triangles ACD, BCD are mutually equilateral : confequently they are alfo mutually equiangular (th. 5), and have the angle ACE equal to the angle BCE.

Hence, the two triangles ACE, BCE, having the two fides AC, CE equal to the two fides BC, CE, and their contained angles equal, are identical (th. 1), and therefore have the fide AE equal to EB. Q. E. D.

## PROBLEM II.

### To Bisect a Given Angle BAC.

FROM the centre A, with any radius, defcribe an arc, cutting off the equal lines AD, AE; and from the two centres D, E, with the fame radius, defcribe arcs interfecting in F; then draw AF, which will bifect the angle A as required.
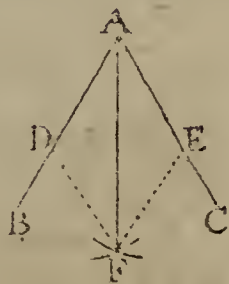
For, join DF, EF. Then the two triangles ADF, AEF, having the two fides AD, DF equal to the two AE, EF (being equal radii), and the fide AF common, are mutually equilateral; confequently they are alfo mutually equiangular (th. 5), and have the angle BAF equal the angle CAF.

*Scholium.* In the fame mannner is a given arc of a circle bifected.

## PROBLEM III.

### At a Given Point C, in a Given Line AB, to Erect a Perpendicular.

From the given point C cut off any equal
parts CD, CE of the given line ; and, from
the two centres D and E, with any one ra-
dius, defcribe arcs interfecting in F ; then
join CF, which will be perpendicular as
required.
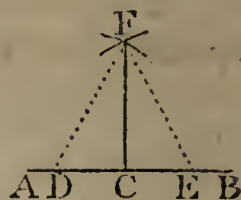
For, draw the two equal radii DF, EF. Then the two
triangles CDF, CEF, having the two fides CD, DF, equal
to the two CE, EF, and CF common, are mutually equi-
lateral ; confequently they are alfo mutually equiangular
(th. 5), and have the two adjacent angles at C equal to each
other ; therefore the line CF is perpendicular to AB
(def. 11).

### Otherwife.

### When the Given Point C is near the End of the Line.

From any point D, affumed above the
line, as a centre, through the given point
C defcribe a circle, cutting the given line
at E ; and through E and the centre D,
draw the diameter EDF ; then join CF,
which will be the perpendicular required.
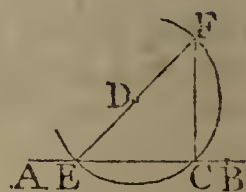
For the angle at C, being an angle in a femircircle, is a
right angle, and therefore the line CF is a perpendicular
(by def. 15).

## PROBLEM IV.

### From a Given Point A, to let fall a Perpendicular on a Given Line BC.

From the given point A as a centre, with
any convenient radius, defcribe an arc, cut-
ting the given line at the two points D and
E ; and from the two centres D, E, with
any radius, defcribe two arcs, interfecting
at F ; then draw AGF, which will be per-
pendicular to BC as required.
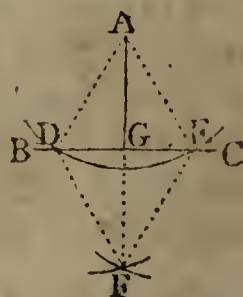
For, draw the equal radii AD, AE, and
DF, EF. Then the two triangles ADF,
                                    AEF,

AEF, having the two fides AD, DF, equal to the two AE, EF, and AF common, are mutually equilateral; confequently they are alfo mutually equiangular (th. 5), and have the angle DAG equal the angle EAG. Hence then, the two triangles ADG, AEG, having the two fides AD, AG, equal to the two AE, AG, and their included angles equal, are therefore equiangular (th. 1), and have the angles at G equal; confequently AG is perpendicular to BC (def. 11).

## Otherwife.

### When the Given point is nearly Oppofite the end of the Line.

FROM any point D, in the given line BC, as a centre, defcribe the arc of a circle through the given point A, cutting BC in E; and from the centre E, with the radius EA, defcribe another arc, cutting the former in F; then draw AGF, which will be perpendicular to BC as required.
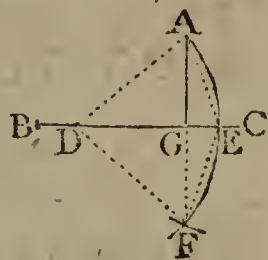
For, draw the equal radii DA, DF, and EA, EF. Then the two triangles DAE, DFE will be mutually equilateral; confequently they are alfo mutually equiangular (th. 5), and have the angles at D equal. Hence, the two triangles DAG, DFG, having the two fides DA, DG, equal to the two DF, DG, and the included angles at D equal, have alfo the angles at G equal (th. 1); confequently thofe angles at G are right angles, and the line AG is perpendicular to DG.

### PROBLEM V.

#### At a Given Point A, in a Given Line AB, to make an Angle Equal to a Given Angle C.

FROM the centres A and C, with any one radius, defcribe the arcs DE, FG. Then, with centre F, and radius DE, defcribe an arc cutting FG in G. Through G draw the line AG, and it will form the angle required.
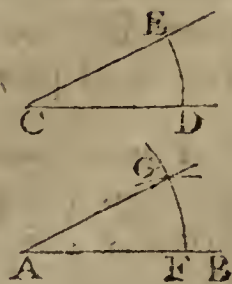
For, conceive the equal lines or radii, DE, FG to be drawn. Then the two tri-
angles

angles CDE, AFG, being mutually equilateral, are mutually equiangular (th. 5), and have the angle at A equal to the angle C.

## PROBLEM VI.

Through a Given Point A, to draw a Line Parallel to a Given Line BC.

FROM the given point A draw the line AD to any point in the given line BC. Then draw the line EAF making the angle at A equal to the angle at D (by prob. 5); fo fhall EF be parallel to BC as required.
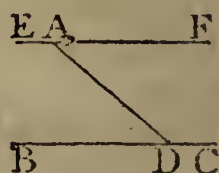
For, the angle D being equal to the alternate angle A, the lines BC, EF, are parallel by th. 13.

## PROBLEM VII.

To Divide a Given Line AB into any propofed Number of Equal Parts.

DRAW any other line AC, forming any angle with the given line AB; on which fet off as many of any equal parts, AD, DE, EF, FC, as the line AB is to be divided into. Join BC; parallel to which draw the other lines FG, EH, DI : then thefe will divide AB in the manner as required.—For thofe parallel lines divide both the fides AB, AC proportionally, by th. 82.
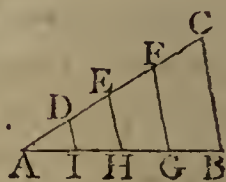
## PROBLEM VIII.

To find a Third Proportional to Two Given Lines AB, AC.

PLACE the two given lines AB, AC forming any angle at A ; and in AB take alfo AD equal to AC. Join BC, and draw DE parallel to it ; fo will AE be the third proportional fought.
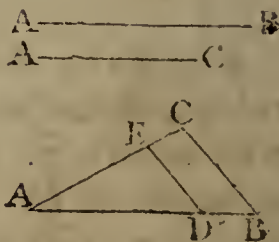
For, becaufe of the parallels BC, DE, the two lines AB, AC are cut proportionally (th. 82); fo that AB : AC :: AD or AC : AE ; therefore AE is the third proportional to AB, AC.

## PROBLEM IX.

To find a Fourth Proportional to three Given Lines AB, AC, AD.

PLACE two of the given lines AB, AC, making any angle at A ; alfo place AD on AB. Join BC ; and parallel to it draw DE : fo fhall AE be the fourth proportional as required.
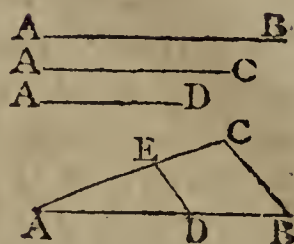
For, becaufe of the parallels BC, DE, the two fides AB, AC are cut proportionally (th. 82) ; fo that AB : AC :: AD : AE.

## PROBLEM X.

To find a Mean Proportional between Two Given Lines AB, BC.

PLACE AB, BC joined in one ftraight line, which bifect in the point O. Then, with the centre O, and radius OA or OC, defcribe the femicircle ADC ; to meet which erect the perpendicular BD, and it will be the mean proportional fought, between AB and BC (by cor. th. 87).
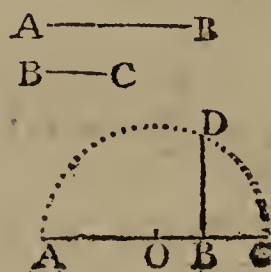
## PROBLEM XI.

To find the Centre of a Given Circle.

DRAW any chord AB ; and bifect it perpendicularly with the line CD, which will be a diameter (th. 41, cor.) Therefore CD bifected in O, will give the centre, as required.
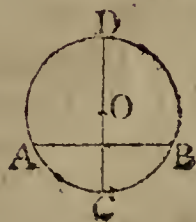
## PROBLEM XII.

To defcribe the Circumference of a Circle through Three Given Points A, B, C.

FROM the middle point B draw chords BA, BC, to the two other points, and bifect thefe chords perpendicularly by lines meeting in O, which will be the centre. Then from the centre O, at the diftance of any one of the points, as OA, defcribe a circle, and it will pafs through the two other points B, C, as required.
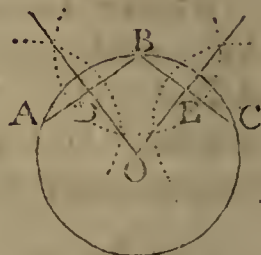
For, the two right-angled triangles OAD, OBD, having the fides AD, DB equal (by conftr.), and OD common, with the included right angles at D equal, have their third fides OA, OB, alfo equal (th. 1). And, in like manner, it is fhewn that OC is equal to OA. So that all the three OA, OB, OC, being equal, will be radii of the fame circle.

## PROBLEM XIII.

To draw a Tangent to a Given Circle, through a Given Point A.

WHEN the given point A is in the circumference of the circle: Join A and the centre O; perpendicular to which draw BAC, and it will be the tangent, by th. 46.
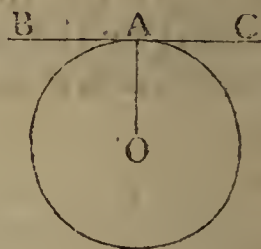
But when the given point A is out of the circle: Draw AO to the centre O; on which as a diameter defcribe a femicircle, cutting the given circumference in D; through which draw BADC, which will be the tangent as required.
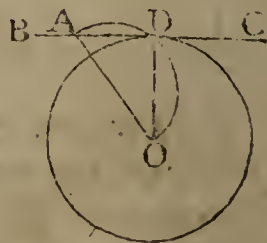
For, join DO. Then the angle ADO, in a femicircle, is a right angle, and confequently AD is a tangent (th. 46).

## PROBLEM XIV.

Upon a Given Line AB to Defcribe a Segment of a Circle, that may Contain a Given Angle C.

At the ends of the given line, make angles DAB, DBA, each equal to the given angle C. Then draw AE, BE perpendicular to AD, BD; and with the centre E, and radius EA or EB, defcribe a circle; fo fhall AFB be the fegment required, as any angle F made in it will be equal to the given angle C.
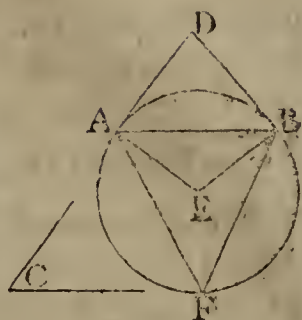
For, the two lines AD, BD, being perpendicular to the radii EA, EB, (by conftr.), are tangents to the circle (th. 46); and the angle A or B, which is equal to the given angle C by conftruction, is equal to the angle F in the alternate fegment AFB (th. 53).

## PROBLEM XV.

To Cut off a Segment from a Given Circle, that fhall Contain a Given Angle C.

Draw any tangent AB to the given circle; and a chord AD to make the angle DAB equal to the given angle C; then DEA will be the fegment required, any angle E made in it being equal to the given angle C.
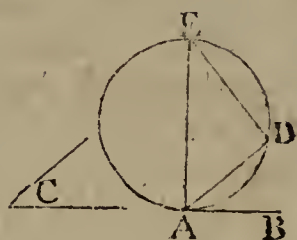
For the angle A, made by the tangent and chord, which is equal the given angle C by conftruction, is alfo equal to any angle E in the alternate fegment (th. 53).

## PROBLEM XVI.

To make an Equilateral Triangle on a Given Line AB.

From the centres A and B, with the diftance AB, defcribe arcs, interfecting in C. Draw AC, BC, and ABC will be the equilateral triangle.

For the equal radii AC, BC, are, each of them, equal to AB.

## PROBLEM XVII.

### To make a Triangle with Three Given Lines AB, AC, BC.

WITH the centre A, and diftance AC, defcribe an arc. With the centre B, and diftance BC, defcribe another arc, cutting the former in C. Draw AC, BC, and ABC will be the triangle required.
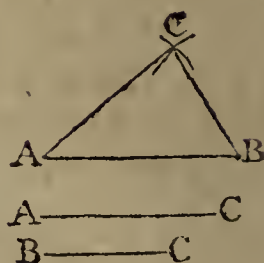
For the radii, or fides of the triangle, AC, BC, are equal to the given lines AC, BC, by conftruction.

## PROBLEM XVIII.

### To make a Square on a Given Line AB.

RAISE AD, BC each perpendicular and equal to AB ; and join DC ; fo fhall ABCD be the fquare fought.
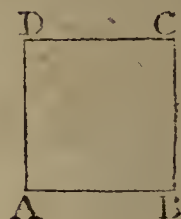
For all the three fides AB, AD, BC are equal, by the conftruction, and DC is equal and parallel to AB (by th. 24) ; fo that all the four fides are equal, and the oppofite ones are parallel. Again, the angle A or B, of the parallelogram, being a right angle, the angles are all right ones (cor. 1, th. 22). Hence then, the figure, having all its fides equal, and all its angles right, is a fquare (def. 34).

## PROBLEM XIX.

### To make a Rectangle, or a Parallelogram, of a Given Length and Breadth, AB, BC.

ERECT AD, BC perpendicular to AB, and each equal to BC ; then join DC, and it is done.
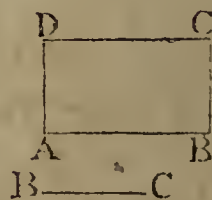
The demonftration is the fame as the laft problem.

And in the fame manner is defcribed any oblique parallelogram, only drawing AD and BC to make the given oblique angle with AB, inftead of perpendicular to it.

## PROBLEM XX.

### To Infcribe a Circle in a Given Triangle ABC.

BISECT the angles at A and B, with the
two lines AD, BD. From the interfec-
tion D, which will be the centre of the
circle, draw the perpendiculars DE, DF,
DG, and they will be the radii of the
circle required:
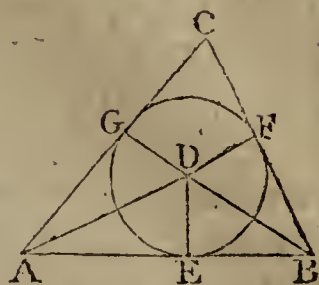
For, fince the angle DAE is equal the
angle DAG, and the angles at E, G right
angles (by conftr.), the two triangles ADE, ADG are equi-
angular ; and, having alfo the fide AD common, they are
identical, and have the fides DE, DG equal (th. 2). In like
manner it is fhewn, that DF is equal to DE or DG.

Therefore, if with the centre D, and diftance DE, a
circle be defcribed, it will pafs through all the three points
E, F, G, in which points alfo it will touch the three fides of
the triangle (th. 46), becaufe the radii DE, DF, DG are
perpendicular to them.


## PROBLEM XXI.

### To Circumfcribe a Circle about a Given Triangle ABC.

BISECT any two fides with two of the
perpendiculars DE, DF, DG, and D will
be the centre.

For, join DA, DB, DC. Then the
two right-angled triangles DAE, DBE
have the two fides DE, EA equal to the
two DE, EB, and the included angles at
E equal, thofe two triangles are therefore
identical (th. 1), and have the fide DA equal DB. In like
manner it is fhewn, that DC is alfo equal to DA or DB. So
that all the three DA, DB, DC, being equal, they are radii
of a circle paffing through A, B, and C.


## PROBLEM XXII.

### To Infcribe an Equilateral Triangle in a Given Circle.

THROUGH the centre C draw any diameter AB. From
the

the point B as a centre, with the radius BC of the given circle, defcribe an arc DCE. Join AD, AE, and ADE is the equilateral triangle fought.

For, join DB, DC, EB, EC. Then DCB is an equilateral triangle, having each fide equal to the radius of the given circle. In like manner, BCE is an equilateral triangle. But the angle ADE is equal the angle CBE, ftanding on the fame arc AE; alfo the angle AED is equal the angle CBD, on the fame arc AD; hence the triangle DAE has two of its angles, ADE, AED, equal to the angles of an equilateral triangle, and therefore the third angle at A is alfo equal to the fame; confequently that triangle is equilateral.

## PROBLEM XXIII.

### To Infcribe a Square in a Given Circle.

DRAW two diameters AC, BD, croffing at right angles in the centre E. Then join the four extremities A, B, C, D, with right lines, and thefe will form the infcribed fquare ABCD.
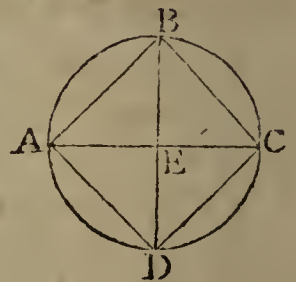
For the four right-angled triangles AEB, BEC, CED, DEA are identical, becaufe they have the fides EA, EB, EC, ED all equal, being radii of the circle, and the four included angles at E all equal, being right angles, by the conftruction. Therefore all their third fides AB, BC, CD, DA are equal to one another, and the figure ABCD is equilateral. Alfo, all its four angles A, B, C, D, are right ones, being angles in a femicircle. Confequently the figure is a fquare.

## PROBLEM XXIV.

### To Circumfcribe a Square about a Given Circle.

DRAW two diameters AC, BD, croffing at right angles in the centre E. Then through the four extremities of them draw FH, IH parallel to AC, and FI, GH parallel to BD, and they will form the fquare FGHI.

For
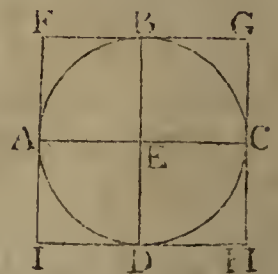
For, the oppofite fides of parallelograms being equal, FG and IH are each equal the diameter AC, and FI and GH each equal the diameter BD; fo that the figure is equilateral. Again, becaufe the oppofite angles of parallelograms are equal, all the four angles F, G, H, I are right angles, being equal to the oppofite angles at E. So that the figure FGHI, having its fides equal, and its angles right ones, is a fquare, and its fides touch the circle at the four points A, B, C, D, being perpendicular to the radii drawn to thofe points.

## PROBLEM XXV.

### To Infcribe a Circle in a Given Square.

BISECT the two fides FG, FI in the points A and B (laft fig.) Then through thefe two points draw AC parallel to FG or IH, and BD parallel to FI or GH. Then the point of interfection E will be the centre, and the four lines EA, EB, EC, ED radii of the infcribed circle.

For, becaufe the four parallelograms EF, EG, EH, EI have their oppofite fides and angles equal : therefore all the four lines EA, EB, EC, ED are equal, being each equal to half a fide of the fquare. So that a circle defcribed from the centre E, with the diftance EA, will pafs through all the points A, B, C, D, and will be infcribed in the fquare, or will touch its four fides in thofe points, becaufe the angles there are right ones.

## PROBLEM XXVI.

### To Circumfcribe a Circle about a Given Square.

DRAW the diagonals AC, BD, and their interfection E will be the centre.

For the diagonals of a fquare bifect each other (th. 40), making EA, EB, EC, ED; all equal, and confequently thefe are radii of a circle paffing through the four points A, B, C, D.
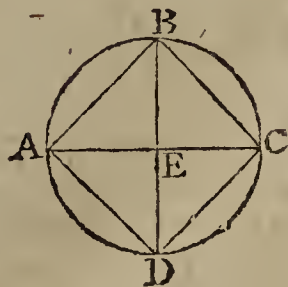
## PROBLEM XXVII.

To Cut a Given Line in Extreme and Mean Proportion.

LET AB be the given line to be divided in extreme and mean ratio, that is, so as that the whole line may be to the greater part, as the greater part is to the lefs part.

Draw BC perpendicular to AB, and equal to half AB. Join AC; and with centre C and diftance CB, defcribe the circle BD; then with centre A and diftance AD, defcribe the arc DE; fo fhall AB be divided in E in extreme and mean ratio, or fo that AB : AE :: AE : EB.
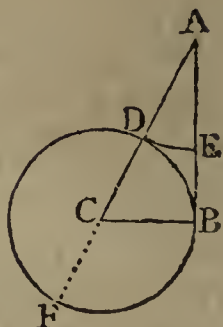
For, produce AC to the circumference at F. Then, ADF being a fecant, and AB a tangent, becaufe B is a right angle: therefore the rectangle AF . AD is equal AB² (cor. 1, th. 61); confequently the means and extremes of thefe are proportional (th. 77), or AB : AF or AD $+$ DF :: AD : AB. But AE is equal AD by conftruction, and AB $=$ 2BC $=$ DF; therefore,          AB : AE $+$ AB :: AE : AB;
and by divifion, AB : AE :: AE : EB.


## PROBLEM XXVIII.

To Infcribe an Ifofceles Triangle in a Given Circle, that fhall have each of the Angles at the Bafe Double the Angle at the Vertex.

DRAW any diameter AB of the given circle; and divide the radius CB, in the point D, in extreme and mean ratio, by the laft problem. From the point B apply the chords BE, BF each equal to CD; then join AE, AF, EF, and AEF will be the triangle required.
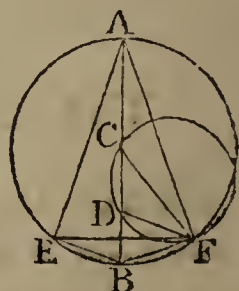
For, through the three points C, D, F defcribe the circle CDF; and draw the lines CF, DF.

Then, becaufe CB : CD :: CD : DB (by conftr.), the rectangle BC . BD $=$ CD² (th. 77) $=$ BF² by conftruction; confequently BF is a tangent to the circle CDF (corol. 1, th. 61). Hence the angle BFD, made by the tangent BF and chord FD, is equal to the angle BCF in the alternate
segment;

fegment; to each of thefe equal angles add the angle DFC, then the whole angle CFB, or its equal CBF (th. 3), is equal to the fum of the two DFC, DCF, which again is equal to the external angle BDF of the triangle CDF (th. 16). Since then the two angles FBD, FDB are equal, their oppofite fides BF, FD are alfo equal (th. 4). But BF, CD are equal (by conftr.) ; therefore CD, DF are equal, being each equal to BF, and confequently their oppofite angles DCF, DFC are alfo equal (th. 3), and the fum of the two together, or the equal external angle BDF, is double of one of them BCF. But BCF is equal to EAF (th. 51), and BDF equal to DBF, or to AEF on the fame arc AF (th. 50); confequently this angle AEF is double of EAF.

Again, the two angles AEB, AFB, being angles in a femicircle, are right angles, and the two fides BE, BA, of the right-angled triangle AEB, are equal to the two fides BF, BA, of the right-angled triangle AFB ; therefore the third fides AE, AF are equal (cor. th. 45). Hence the triangle AEF is ifofceles, and has each angle at the bafe double the angle at the vertex.

<br>

<center>PROBLEM XXIX.</center>

<br>

<center>To Infcribe a Regular Pentagon in a Given Circle.</center>

INSCRIBE the ifofceles triangle ABC having each of the angles ABC, ACB double the angle BAC (prob. 28). Then bifect the two arcs ADB, AEC, in the points D, E ; and draw the chords AD, DB, AE, EC, fo fhall ADBCE be the infcribed equilateral pentagon required.
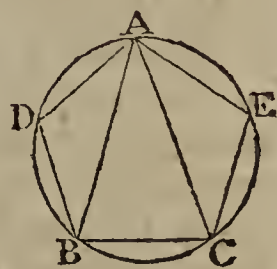
For, becaufe equal angles ftand on equal arcs, and double angles on double arcs, alfo the angles ABC, ACB being each double the angle BAC, therefore the arcs ADB, AEC, fubtending the two former angles, are each double the arc BC fubtending the latter. And fince the two former arcs are bifected in D and E, it follows that all the five arcs AD, DB, BC, CE, EA are equal to each other, and confequently the chords alfo which fubtend them, or the five fides of the pentagon, are all equal.

<center>B b 2</center>

PROBLEM XXX.

To Inscribe a Regular Hexagon in a Given Circle.

APPLY the radius of the given circle AO, as a chord AB, and it will be a side of the regular hexagon ABCDEF.

For, draw the radii AO, BO, CO, DO, EO, FO. Then the triangle ABO being equilateral (by conftr.), its three angles are all equal (cor. 2, th. 3), and any one of them, as AOB, is one-third of two right angles (th. 17), or one-sixth of four right angles. But the whole circumference is the meafure of four right angles (cor. 4, th. 6). Therefore the arc AB is one-sixth of the circumference of the circle, and confequently its chord AB one fide of an equilateral hexagon infcribed in the circle.

*Corol.* The fide of a regular hexagon is equal to the radius of the circumfcribing circle, or to the chord of one-sixth part of the circumference.

PROBLEM XXXI.

To defcribe a Regular Pentagon or Hexagon about a Given Circle.

IN the given circle infcribe a regular polygon of the fame name or number of fides, as ABCDE, by one of the foregoing problems. Then to all the angular points of it draw tangents (by prob. 13), and thefe will form the circumfcribing polygon required.
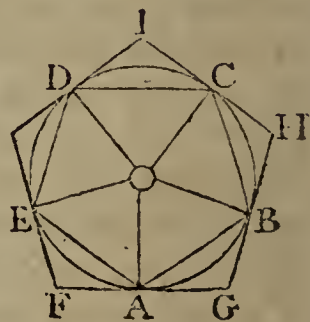
For, all the chords, or fides of the infcribing figure, AB, BC, &c, being equal, and all the radii OA, OB, &c, being equal, all the vertical angles about the point O are equal. But the angles OEF, OAF, OAG, OBG, made by the tangents and radii, are right angles; therefore OEF + OAF = two right angles, and OAG + OBG = two right angles; confequently, alfo, AOE + AFE = two right angles, and AOB + AGB = two right angles (cor. 2, th. 18). Hence, then, the angles AOE + AFE being = AOB + AGB, of which AOB is = AOE:

AOE ; confequently the remaining angles F and G are alfo equal. In the fame manner it is fhewn, that all the angles F, G, H, I, K are equal.

Again, the tangents from the fame point FE, FA are equal, as alfo the tangents AG, GB (th. 61, cor. 2) ; alfo, the angles F and G of the ifofceles triangles AFE, AGB, are equal, as well as their oppofite fides AE, AB ; confequently thofe two triangles are identical (th. 1), and have their other fides EF, FA, AG, GB all equal, and FG equal to the double of any one of them. In like manner it is fhewn that all the other fides GH, HI, IK, KF are equal to FG, or double of the tangents GB, BH, &c.

Hence, then, the circumfcribed figure is both equilateral and equiangular, which was to be fhewn.

*Corol.* The infcribed circle touches the middles of the fides of the polygon.

## PROBLEM XXXII.

### To Infcribe a Circle in a Regular Polygon.

BISECT any two fides of the polygon by the perpendiculars GO, FO, and their interfection O will be the centre of the infcribed circle, and OG or OF will be the radius.



For the perpendiculars to the tangents AF, AG, pafs through the centre (cor. th. 47) ; and the infcribed circle touches the middle points F, G, by the laft corollary. Alfo, the two fides AG, AO, of the right-angled triangle AOG, being equal to the two fides AF, AO, of the right-angled triangle AOF, the third fides OF, OG will alfo be equal (cor. th. 45). Therefore the circle defcribed with the centre O and radius OG, will pafs through F, and will touch the fides in the points G and F. And the fame for all the other fides of the figure.
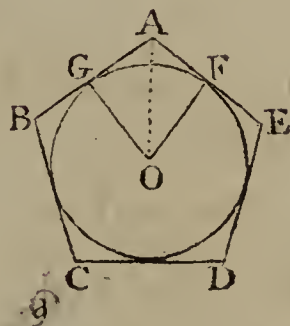
PROBLEM XXXIII.

To defcribe a Circle about a Regular Polygon.

BISECT any two of the angles, C and D, with the lines CO, DO ; then their inter-fection O will be the centre of the cir-cumfcribing circle; and OC, or OD, the radius.
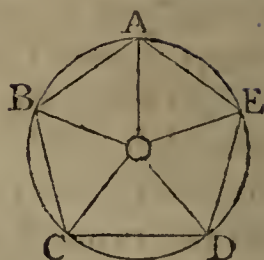
For, draw OB, OA, OE, &c, to the angular points of the given polygon. Then the triangle OCD is ifofceles, having the angles at C and D equal, being the halves of the equal angles of the polygon BCD; CDE ; therefore their oppofite fides CO, DO are equal (th. 4). But the two triangles OCD, OCB, having the two fides OC, CD equal to the two OC, CB, and the included angles OCD, OCB alfo equal, will be identical (th. 1), and have their third fides BO, OD equal. In like manner it is fhewn, that all the lines OA, OB, OC, OD, OE are equal. Confequently a circle defcribed with the centre O and radius OA, will pafs through all the other angular points, B, C, D, &c, and will circumfcribe the polygon.

PROBLEM XXXIV.

To make a Square Equal to the Sum of two Given Squares.

LET AB and AC be the fides of the two given fquares. Draw two indefinite lines AP, AQ at right angles to each other ; in which place the fides AB, AC, of the given fquares ; join BC ; then a fquare defcribed on BC will be equal to the fum of the two fquares defcribed on AB and AC (th. 34).
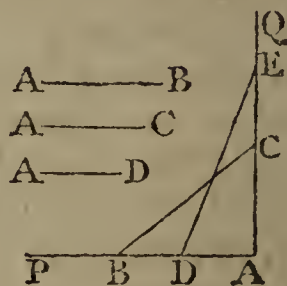
*Scholium.* In the fame manner, a fquare may be made equal to the fum of three or more given fquares. For, if AB, AC, AD be taken as the fides of the given fquares, then, making AE = BC, AD = AD, and drawing DE, it is evident that the fquare on DE will be equal to the fum of the three fquares on AB, AC, AD. And fo on for more fquares.

3

## PROBLEM XXXV.

### To make a Square Equal to the Difference of two Given Squares.

Let AB and AC, taken in the same straight line, be equal to the sides of the two given squares.—From the centre A, with the distance AB, describe a circle; and make CD perpendicular to AB, meeting the circumference in D : so shall a square described on CD be equal to $AD^2 - AC^2$ or $AB^2 - AC^2$, as required (cor. th. 34).
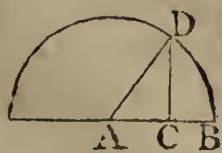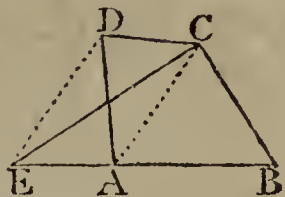
## PROBLEM XXXVI.

### To make a Triangle Equal to a Given Quadrilateral ABCD.

Draw the diagonal AC, and parallel to it DE, meeting BA produced at E, and join CE : then will the triangle CEB be equal to the given quadrilateral ABCD.

For, the two triangles ACE, ACD, being on the same base AC, and between the same parallels AC, DE, are equal (th. 25); therefore, if ABC be added to each, BCE will be equal to ABCD (ax. 2).

## PROBLEM XXXVII.

### To make a Triangle Equal to a Given Pentagon ABCDE.

Draw DA and DB, and also EF, CG parallel to them, meeting AB produced at F and G ; then draw DF and DG; so shall the triangle DFG be equal to the given pentagon ABCDE.

For the triangle DFA = DEA, and the triangle DGB = DCB (th. 25); therefore, by adding DAB to the equals, the sums are equal (ax. 2), that is, DAB + DAF + DBG = DAB + DAE + DBC, or the triangle DFG = the pentagon ABCDE.
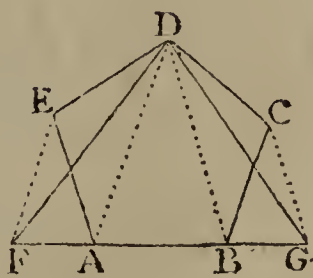
## PROBLEM XXXVIII.

### To make a Rectangle Equal to a Given Triangle ABC.

BISECT the bafe AB in D; then raife
DE and BF perpendicular to AB, and
meeting CF, parallel to AB, at E and F:
fo fhall DF be the rectangle equal to the
given triangle ABC (by cor. 2, th. 26).
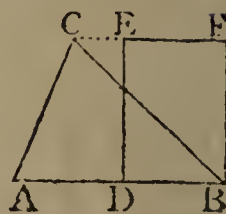
## PROBLEM XXXIX.

### To make a Square Equal to a Given Rectangle ABCD.

PRODUCE one fide AB, till BE be
equal to the other fide BC. On AE
as a diameter defcribe a circle, meeting
BC produced at F: then will BF be the
fide of the fquare BFGH, equal to the
given rectangle BD, as required, as ap-
pears by Cor. Th. 87, and Th. 77.
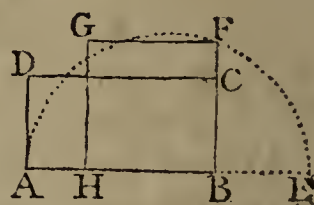
APPLI-

# APPLICATION of ALGEBRA

## TO

# GEOMETRY.

WHEN it is propofed to refolve a geometrical problem algebraically, or by algebra, it is proper, in the firft place, to draw a figure that fhall reprefent the feveral parts or conditions of the problem, and to fuppofe that figure to be the true one. Then, having confidered attentively the nature of the problem, the figure is next to be prepared for a folution, if neceffary, by producing or drawing fuch lines in it, as appear moft conducive to that end. This done, the ufual fymbols or letters, for known and unknown quantities, are employed to denote the given parts of the figure, or as many of them as neceffary, as alfo fuch unknown line or lines as may be eafieft found, whether required or not. Then proceed to the operation, by obferving the relations that the feveral parts of the figure have to each other ; from which, and the proper theorems in the foregoing elements of geometry, make out as many equations, independent of each other, as there are unknown quantities, employed in them : the refolution of which equations, in the fame manner as in arithmetical problems, will determine the unknown quantities, and refolve the problem propofed.

As no general rule can be given for drawing the lines, and felecting the fitteft quantities to fubftitute for, fo as always to bring out the moft fimple conclufions, becaufe different problems require different methods of folution ; the beft way to gain experience in this matter, is to try the folution of the fame problem in different ways, and then apply that which fucceeds beft, to other cafes of the fame kind, when they afterwards occur. The following particular directions, however, may be of fome ufe in this matter.

1ft, In preparing the figure, by drawing lines, let them be either parallel or perpendicular to other lines in the figure, or fo as to form fimilar triangles. And if an angle be given, it will be proper to let the perpendicular be oppofite to that angle, and to fall from one end of a given line, if poffible.

2nd,

2*nd*, In felecting the quantities proper to fubftitute for, thofe are to be chofen, whether required or not, which lie neareft the known or given parts of the figure, and by means of which the next adjacent parts may be expreffed by addition and fubtraction only, without the intervention of furds.

3*d*, When two lines or quantities are alike related to other parts of the figure or problem, the beft way is, not to make ufe of either of them feparately, but to fubftitute for their fum, or difference, or rectangle, or the fum of their alternate quotients, or for fome line or lines, in the figure, to which they have both the fame relation.

4*th*, When the area, or the perimeter, of a figure, is given, or fuch parts of it as have only a remote relation to the parts required; it is fometimes of ufe to affume another figure fimilar to the propofed one, having one fide equal to unity, or fome other known quantity. For, from hence the other parts of the figure may be found, by the known proportions of the like fides, or parts, and fo an equation be obtained.

For examples of this matter take the following problems.

## PROBLEM I.

*In a Right-angled Triangle, having given the Bafe* (3), *and the Sum of the Hypothenufe and Perpendicular* (9); *to find both thefe two Sides.*

Let ABC reprefent the propofed triangle, right-angled at B. Put the bafe $AB = 3 = b$, and the fum $AC + BC$ of the hypothenufe and perpendicular $= 9 = s$; alfo, let $x$ denote the perpendicular BC; hence it follows that the hypothenufe AC will be expreffed by $s - x$.
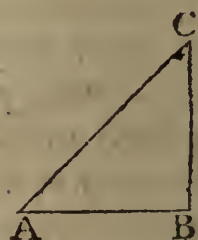
But $AC^2 = AB^2 + BC^2$ (by theor. 34. Geom ); that is, $(s-x)^2 = b^2 + x^2$, or $s^2 - 2sx + x^2 = b^2 + x^2$, or $s^2 - b^2 = 2sx$; hence $x = \dfrac{s^2 - b^2}{2s} = 4 = BC$, and $s - x = \dfrac{s^2 + b^2}{2s} = 5 = AC$, the perpendicular and hypothenufe as required.

## PROBLEM II.

*In a Right-angled Triangle, having given the Hypothenuse* (5); *and the Sum of the Bafe and Perpendicular* (7); *to find both thefe two Sides.*

LET ABC reprefent the propofed triangle, right-angled at B. Put the given hypothenufe $AC = 5 = a$, and the fum $AB + BC$ of the bafe and perpendicular $= 7 = s$; alfo, let $x$ denote the bafe AB; hence it follows that the perpendicular BC will be expreffed by $s - x$.

But $AC^2 = AB^2 + BC^2$ by the nature of right-angled triangles; that is, $a^2 = x^2 + (s - x)^2 = x^2 + s^2 - 2sx + x^2$, or $2x^2 - 2sx = a^2 - s^2$, or $x^2 - sx = \frac{1}{2}a^2 - \frac{1}{2}s^2$; hence, by completing the fquare, &c, is found $x = \frac{1}{2}s - \frac{1}{2}\sqrt{2a^2 - s^2} = 3 = AB$; and $s - x = \frac{1}{2}s + \frac{1}{2}\sqrt{2a^2 - s^2} = 4 = BC$; the bafe and perpendicular required.

### Otherwife.

LET $z$ denote the difference of the bafe and perpendicular, the fum being denoted by $s$, as above. Then, becaufe that, in any two quantities, half their difference added to half the fum gives the greater quantity, and the fame fub-tracted gives the lefs quantity (by Ex. 2, Alg. pag. 243), it follows that $\frac{1}{2}s + \frac{1}{2}z$ and $\frac{1}{2}s - \frac{1}{2}z$ will denote the two legs, or the perpendicular and bafe. Hence then $a^2 = (\frac{1}{2}s + \frac{1}{2}z)^2 + (\frac{1}{2}s - \frac{1}{2}z)^2 = \frac{1}{2}s^2 + \frac{1}{2}z^2$; or $z^2 = 2a^2 - s^2$, and $z = \sqrt{2a^2 - s^2}$. Then the half of this being added and fub-tracted with $\frac{1}{2}s$ the half fum, gives $- - - - - - -$ $\frac{1}{2}s + \frac{1}{2}z = \frac{1}{2}s + \frac{1}{2}\sqrt{2a^2 - s^2}$, and $\frac{1}{2}s - \frac{1}{2}z = \frac{1}{2}s - \frac{1}{2}\sqrt{2a^3 - s^2}$, the fame as before.

## PROBLEM III.

*In a Rectangle, having given the Diagonal* (10), *and the Perimeter, or Sum of all the Four Sides* (28); *to find each of the Sides feverally.*

LET ABCD be the propofed rectangle; and put the diagonal $AC = 10 = d$, and half the perimeter $AB + BC$ or $AD + DC = 14 = a$; alfo put one fide $AB = x$; then the other fide BC will be denoted by $a - x$. Hence, by right-angled triangles,
$$AC^2$$
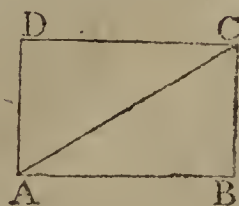
$AC^2 = AB^2 + BC^2$, that is, $d^2 = x^2 + (a-x)^2 = 2x^2 - 2ax + a^2$ : confeq. $x^2 - ax = \frac{1}{2}d^2 - \frac{1}{2}a^2$, and by completing the fquare, &c, $x = \frac{1}{2}a + \frac{1}{2}\sqrt{2d^2 - a^2} = 8 = AB$, and $a - x = \frac{1}{2}a - \frac{1}{2}\sqrt{2d^2 - a^2} = 6 = BC$, the two fides; and the fame for the other two AD, DC.

Or, by fubftituting for the fum and difference of the two fides, AB, BC, this problem may be refolved, like the latter method of the foregoing one, by a fimple quadratic, and confequently without completing the fquare.

<center>PROBLEM IV.</center>

*Having given the Bafe and Perpendicular of any Triangle; to find the Side of a Square Infcribed in the fame..*

LET ABC reprefent the given triangle, and EFGH its infcribed fquare. Put the bafe $AB = b$, the perpendicular $CD = a$, and the fide of the fquare GF or GH $=$ $DI = x$; then will $CI = CD - DI = a - x$.
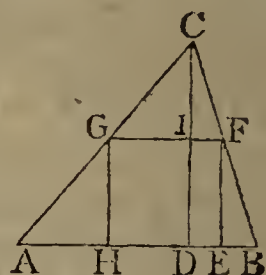
Then, becaufe the like lines in the fimilar triangles ABC, GFC, are proportional, (by theor. 84, Geom.), AB : CD :: GF : CI, that is, $b : a :: x : a - x$. Hence $ab - bx = ax$, or $- - -$ $ab = ax + bx$, and confequently $x = \dfrac{ab}{a+b} = GF$ or GH the fide of the infcribed fquare; which therefore is of the fame magnitude, whatever the fpecies or the angles of the triangle may be.

<center>PROBLEM V.</center>

*In an Equilateral Triangle, having given the lengths of the three Perpendiculars, drawn from a certain Point within, on the three Sides; to determine the Sides.*

LET ABC reprefent the equilateral triangle, and DE, DF, DG the given perpendiculars from the point D. Draw the lines DA, DB, DC to the three angular points : and let fall the perpendicular CH on the bafe AB. Put the three given perpendiculars $DE = a$, $DF = b$, $DG = c$, and put $x = AH$ or BH, half the fide of the equilateral triangle. Then is AC or $BC = 2x$, and by right-angled triangles the perpendicular $CH = \sqrt{AC^2 - AH^2} = \sqrt{4x^2 - x^2} = \sqrt{3x^2} = x\sqrt{3}$.

<div align="right">Now,</div>
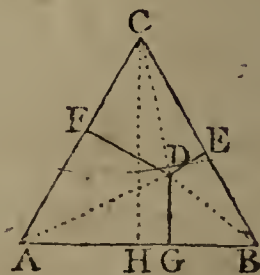
Now, fince the area or fpace of a rectangle is expreffed by the product of the bafe and height (cor. 2, th. 81, Geom.), and that a triangle is equal to half a rectangle of equal bafe and height (cor. 1, th. 26), it follows that,

the whole triangle ABC is $= \frac{1}{2}AB \times CH = x \times x\sqrt{3} = x^2\sqrt{3}$,
the triangle ABD $= \frac{1}{2}AB \times DG = x \times c = cx$,
the triangle BCD $= \frac{1}{2}BC \times DE = x \times a = ax$,
the triangle ACD $= \frac{1}{2}AC \times DF = x \times b = bx$.

But the three laft triangles make up, or are equal to, the whole former or great triangle;

that is, $x^2\sqrt{3} = ax + bx + cx$; hence then

$x = \dfrac{a + b + c}{\sqrt{3}}$, half the fide of the triangle fought.

Alfo, fince the whole perpendicular CH is $= x\sqrt{3}$, it is therefore $= a + b + c$. That is, the whole perpendicular CH, is juft equal to the fum of all the three fmaller perpendiculars DE + DF + DG taken together, wherever the point D is fituated.

### PROBLEM VI.

In a Right-angled Triangle, having given the Bafe (3), and the Difference between the Hypothenufe and Perpendicular (1); to find both thefe two Sides.

### PROBLEM VII.

In a Right-angled Triangle, having given the Hypothenufe (5), and the Difference between the Bafe and Perpendicular (1); to determine both thefe two Sides.

### PROBLEM VIII.

Having given the Area, or Meafure of the Space, of a Rectangle, infcribed in a given Triangle; to determine the Sides of the Rectangle.

### PROBLEM IX.

In a Triangle, having given the Ratio of the two Sides, together with both the Segments of the Bafe, made by a Perpendicular from the Vertical Angle; to determine the Sides of the Triangle.

### PROBLEM X.

In a Triangle, having given the Bafe, the Sum of the other two Sides, and the Length of a Line drawn from the Vertical Angle to the Middle of the Bafe; to find the fides of the Triangle.

### PROBLEM XI.

In a Triangle, having given the two Sides, about the Vertical Angle, and the Line bisecting that Angle, and terminating in the Base; to find the Base.

### PROBLEM XII.

To determine the Radii of three Equal Circles, described in a given Circle, to touch each other and also the Circumference of the given Circle.

### PROBLEM XIII.

In a right-angled Triangle, having given the Perimeter or Sum of all the Sides, and the Perpendicular let fall from the Right Angle on the Hypothenuse; to determine the Triangle, that is, its Sides.

### PROBLEM XIV.

To determine a Right-angled Triangle; having given the Perimeter, and the Radius of its Inscribed Circle.

### PROBLEM XV.

To determine a Right-angled Triangle; having given the Lengths of two Lines drawn from the acute Angles, to the Middle of the opposite Sides.

### PROBLEM XVI.

To determine a Right-angled Triangle; having given the Hypothenuse, and the Difference of two Lines drawn from the two acute Angles to the Centre of the Inscribed Circle.

### PROBLEM XVII.

To determine a Triangle; having given the Base, the Perpendicular, and the Ratio of the two Sides.

### PROBLEM XVIII.

To determine a Right-angled Triangle; having given the Hypothenuse, and the Side of the Inscribed Square.

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PROBLEM XIX.

To determine a Triangle; having given the Bafe, the Perpendicular, and the Difference of the two other Sides.

PROBLEM XX.

To determine a Triangle; having given the Bafe, the Perpendicular, and the Rectangle or Product of the two Sides.

PROBLEM XXI.

To determine a Triangle; having given the Lengths of three Lines drawn from the three Angles, to the Middle of the oppofite Sides.

PROBLEM XXII.

In a Triangle, having given all the three Sides; to find the Radius of the Infcribed Circle.

PROBLEM XXIII.

To determine a Right-angled Triangle; having given the Side of the Infcribed Square, and the Radius of the In-fcribed Circle.

PROBLEM XXIV.

To determine a Triangle, and the Radius of the Infcribed Circle; having given the Lengths of three Lines drawn from the three Angles, to the Centre of that Circle.

PROBLEM XXV.

To determine a Right-angled Triangle; having given the Hypothenufe, and the Radius of the Infcribed Circle.

PROBLEM XXVI.

To determine a Triangle; having given the Bafe, the Line bifecting the Vertical Angle, and the Diameter of the Circumfcribing Circle.

## END of VOL. I.

# ERRATA.

Page 48, The Anſwer to Queſtion 40, *read* 33 1l. 1s. 9¼d.

    51, Line 12, *for* 288 $\frac{59}{207}$ *read* 278 $\frac{204}{207}$.

    88, Lines 9 and 10, *for* 834 *read* 560.

    129, Anſ. to Queſtion 8, *read* 10l. 3s. 4½d.

    165. Ex. 26, *for* $ab\frac{2}{3}$ *read* $\frac{2}{3}ab$.